PROGRAM FOR EVALUATION OF SIGNIFICANCE, CONFIDENCE INTERVALS AND LIMITS BY DIRECT CALCULATION OF PROBABILITIES

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We propose a program which allows one to estimate significance, confidence intervals and limits, taking into account systematics and statistical uncertainties of variables described by Poisson distributions. The given program can be used for combining searches. The motivation of the direct probabilities calculations is determined by two reasons. Firstly, the tail of a Poisson distribution is heavier than that of a normal distribution. In the case of small probabilities the Gaussian approximation gives the wrong values of estimators. Secondly, the estimators which are constructed on the basis of the likelihood ratio often have poor statistical properties.

1. Introduction

In searches for new particles in high energy physics the tasks of the comparison of the possibilities of different planned experiments and of the setting and combining of the results of running experiments are vitally important. The determination of the significance of the results and the setting of confidence limits are related to these tasks.

There are many reviews about using of the notion of a significance in various areas of physics (for example, [1–3]). According to ref. [4] "Common practice is to express the significance of an enhancement by quoting the number of standard deviations". In the case of asymmetric distributions this statement is not clear. Then the significance can be quoted in terms of equivalent standard deviations of the normal distribution.

The definition chosen for significance depends on the goal. The simplest significance $S_1 = \dfrac{s}{\sqrt{b}}$ (or Z_{sb} in the notation of refs. [2]) takes into account only the fluctuation of the background with the assumption that the background obeys the normal distribution with mean and variance equals to b. The S_1 corresponds the case when we observed $s + b$ events. The significance $S_2 = \dfrac{s}{\sqrt{s+b}}$ takes into account the fluctuation of the normal distributed random value *signal+background* with the expected background b. The significance $S_{12} = \sqrt{s+b} - \sqrt{b}$ [5] takes into account the fluctuations of the signal and of the background with the assumption that the signal and the background obey the normal distribution with mean and variance which equal s for signal and b for background.

In the case of the asymmetrically distributed signal and background these significances can be used only as approximations [6]. Thus for Poisson distributions, the significance S_{cP} [7] was proposed as the analogy of S_1. The significance S_{cP} is determined by direct calculations of probabilities.

The correct choice of the confidence interval for the parameter of the distribution under study also is a hot problem of data analysis. This matter was discussed in many Workshops and Conferences [8]. There are many methods of constructing intervals: credible, tolerant, fiducial. Physicists [9] formulated the problem of the specialist in data analysis "the only remaining problem: make a choice ... chosen method should be as simple as possible, but not wrong". In many cases the reconstruction of the confidence density [10] of the parameter (by direct calculation of probabilities) is the solution of this problem. The knowledge of the confidence density allows to determine any confidence intervals (central interval, shortest interval, interval with optimal coverage and so on) by the unique way and to take into account the systematics and statistical uncertainties.

2. Significance

The significance S_{cP} is the probability from Poisson distribution with mean b to observe $b+s$ or more events, converted to the equivalent number of sigmas of a Gaussian distribution, i.e.

$$\beta = \frac{1}{\sqrt{2\pi}} \int_{S_{cP}}^{\infty} e^{-\frac{x^2}{2}} dx, \texttt{ where } \beta = \sum_{i=s+b}^{\infty} \frac{b^i e^{-b}}{i!}. \tag{1}$$

The program *ScP* allows one to calculate the significance defined in a such way. The background uncertainties are incorporated into the program. The program takes into account two types of uncertainties: the systematic uncertainty with statistical properties (the normal distribution with mean which equals 0 and the variance – σ_b^2) and the uncertainty without statistical properties [5] (theoretical uncertainty with bias in the background $b + \delta_b$ and conserving the scale, i.e. $s+b$ without bias). Also, the program allows one to combine several observed values of significance. The values of S_{cP} by definition have the irregular behavior for non-integer $s+b$. The program has the option which allows the smoothing of the result by the using of "the continuous Poisson distribution" [11].

3. Confidence Limits

The knowledge of the confidence density allows one to determine any confidence limits for the parameter of the distribution. The program *Limsb* is the realization of this idea. The Gamma-distributions $\Gamma_{1,\hat{n}+1}$ is statistically dual to the Poisson distribution in the case of $\hat{n}$ observed events from the Poisson flow of events [10, 11]. It means that the direct calculations of the probability density of the Gamma-distribution $\Gamma_{1,\hat{n}+1}$ are the reconstruction of the confidence density of the unknown parameter of the Poisson distribution. The confidence intervals produced by the program *Limsb* coincide with the corresponding confidence intervals calculated in the framework of the Bayesian approach with a uniform prior.

4. Conclusions

All significances described in Sec. 1 relate to the measurement of a random variable. However, significance S_{cP} can be generalized for the determination of the significance of the estimated parameter by the use of confidence densities. We plan to include the presented approach for determination of the significance and confidence limits into the ROOT system [12]. The programs *ScP* and *Limsb* can be found in Web page http://cmsdoc.cern.ch/˜bityukov.

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