A NEW FAST TRACK-FIT ALGORITHM BASED ON BROKEN LINES

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The determination of the particle momentum in HEP experiments requires a fit of a parametrization to the points measured in a tracking chamber. A new non-recursive track-fit algorithm based on broken lines allows the reconstruction of the particle trajectory taking into account details of the multiple scattering. It provides optimal parameters and their covariance matrices at track start and end, and optimal values at each measured point along the trajectory including the variances. The parametrization of the trajectory allows the use of sparse-matrix techniques with a total execution time $\mathcal{O}(n)$, and the new algorithm is, under test conditions, a factor six faster than the Kalman filter.

1. Track measurement in particle physics

For a HEP tracking detector with a homogeneous magnetic field $\boldsymbol{B}_z$ (in z-direction) the *ideal track parametrization* is a *helix* with five parameters: the curvature κ (inverse radius, signed), the distance d_{ca} and the angle ϕ_0 at the point of closest approach to the axis, the intercept z_0 and the slope parameter $\tan\lambda \equiv \cot\vartheta$. Various effects can result in deviations to the ideal helix curve, and the track fit with a pure helix parametrization is not optimal. Multiple scattering deflections will influence all downstream measurement in a correlated way, and delimit the accuracy of momentum measurement at low momenta. There are effects of the field inhomogeneity and continuous energy loss along the trajectory (radiation in case of electrons).

Different methods of track fitting[2] exist which are able to take the effects mentioned above into account. In global methods with a computing time $\propto n^3$, where n is the number of data points, the track parameters are determined in a single step. In the *matrix method* all effects of multiple scattering are included in the covariance matrix of the measured points, which becomes non-diagonal; in the *breakpoint method* a certain number of scattering planes is defined, increasing the number of parameters, while the covariance matrix of the measured points remains diagonal. In the *progressive method*[3] the track is followed by incorporating measurement after measurement with update of the parameter vector and covariance matrix, starting from the outer detector. The method is equivalent to the *Kalman filter*, which became the standard method of track fitting; multiple scattering is introduced as process noise and optimal track parameters are determined at both ends of the track by smoothing in the direction opposite to the filter. These methods have a computing time $\propto n$ and are faster by a large factor compared with the global methods mentioned before.

The method proposed here can be considered as a global method too. First approximate track parameters are determined from simple 2D fits of a circle

$$\frac{1}{2}\kappa\left(x_i^2+y_i^2+d_{\text{ca}}^2\right) - \left(1+\kappa d_{\text{ca}}\right)\left(x_i\sin\phi_0 - y_i\cos\phi_0\right) + d_{\text{ca}} = 0$$

and of a straight line $z_i = z_0 + (\tan\lambda)\cdot s_i$ to the data. Then residuals w.r.t. the circle and the straight line are calculated as a function of the track length s_i (in the $r\phi$-plane); a *detailed fit to the residuals* taking into account multiple scattering (and perhaps other effects) is made, where corrections to the track parameters like $\Delta\kappa$ are determined. Due to the special parametrization of the trajectory in the residual fit the computing time is $\propto n$ and the method is, under test conditions, faster by a factor of six, compared with the Kalman filter and smoothing. Results for the track parameters are almost identical for the global and Kalman methods. An example for a track fit is shown in Figure 3. A circle fit is shown on the left, and the residuals to the circle are shown on the right as a function of the track length; the fit of the residuals is discussed in later sections.

2. Multiple scattering

A charged particle traversing material will make a large number of small angle collisions, called multiple scattering, which is dominated by Coulomb scattering off the nuclei. Multiple scattering is parametrized by two mutually orthogonal, uncorrelated angles.

The Review of Particle Properties PDG[1] quotes the formula

$$V[\theta] = \theta_0^2 = \left(\frac{13.6\ \mathrm{MeV}}{\beta p c}\right)^2 t\ [1 + 0.038 \ln t]^2 \quad (1)$$

for the variance of the deflection angle θ of a singly charged particle with momentum p and velocity β. The quantity t is the thickness of the material in units of the radiation length X_0, thus $t = \Delta s / X_0$.

The trajectory of a charged particle traversing a *homogeneous* medium of thickness Δs between two detector planes is shown in Figure 1. The effect of multiple scattering after traversal of a homogeneous medium of thickness Δs can be described by two parameters, e.g. the *deflection angle* θ_{plane}, or just θ, and the *angle* ψ, which is the angle between the original particle direction and the straight line between the two intersection points (circles). The two angles are statistically correlated. In an ideal detector

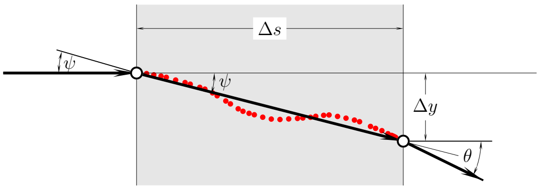

Fig. 1. Quantities used to describe multiple scattering.

the intersections could be measured with high precision and the straight line between the two intersections gives the complete information on the particle trajectory, which is available from the measurement. The angle between the direction of this line and the true particle direction is $\psi_{\text{left}} \equiv \psi$ on the left and $\psi_{\text{right}} \equiv \theta - \psi$ on the right of the medium. Expectation and variance of these two angles are identical with

$$E\begin{bmatrix}\psi_{\text{left}}\\ \psi_{\text{right}}\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad V\begin{bmatrix}\psi_{\text{left}}\\ \psi_{\text{right}}\end{bmatrix} = \begin{pmatrix}1/3 & 1/6\\ 1/6 & 1/3\end{pmatrix}\theta_0^2 \quad (2)$$

for a homogeneous medium between the two detector planes. Usually the material distribution between two detector planes is inhomogeneous and the covariance matrix has to be calculated from the geometry of the material distribution within the layer i:

$$\boldsymbol{V}\begin{bmatrix}\psi_{\text{left}}\\ \psi_{\text{right}}\end{bmatrix}_i = \begin{pmatrix}V_{L,i} & V_{LR,i}\\ V_{LR,i} & V_{R,i}\end{pmatrix}, \quad (3)$$

where the matrix elements are proportional to the value of θ_0^2, calculated from t by the formula (1).

3. Tracking in the sz-plane

The approximate value of the momentum p determined in the simple 2D fits allows the calculation of the multiple scattering variances $V[\theta]$. In the fit of the residuals in the sz-plane corrections to the parameters z_0 and $\tan\lambda$ are determined, taking multiple scattering into account.

Figure 2 shows the trajectory of a charged particle with multiple scattering, and the intersection points of the trajectory with the detector planes. The coordinates y_i, with standard deviation σ_i, represent the residuals of the measurements at the detector planes with coordinates s_i, for $i = 1, 2, \ldots n$; they are transverse to the average track, and are uncorrelated. For an improved fit taking into account the multiple scattering effects, the new track-fit method developed here uses two phases in the track reconstruction:

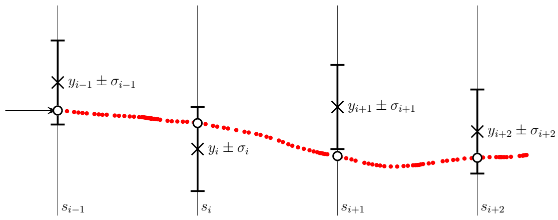

Fig. 2. Particle trajectory and measured residuals y_i.

Reconstruction of the trajectory: The trajectory, represented by the intersection points of the trajectory with the detector planes, is determined in a least squares fit; the estimates of the intersection points are denoted by u_i.

Track parameter determination: From the fitted u_i-values the two track parameters intercept and slope, required for the physics analysis, are determined at both sides of the track.

The least squares fit thus has n parameters u_i to be determined, one for each measured value y_i. The sum of squares to be minimized includes the sum of $(y_i - u_i)^2/\sigma_i^2$. The intersection points u_i to be fitted are shown in Figure 4. Each pair of adjacent points are connected by a straight line, and due to multiple scattering there is a kink angle,

$$\beta_i = \psi_{\text{right},i-1} - \psi_{\text{left},i}$$

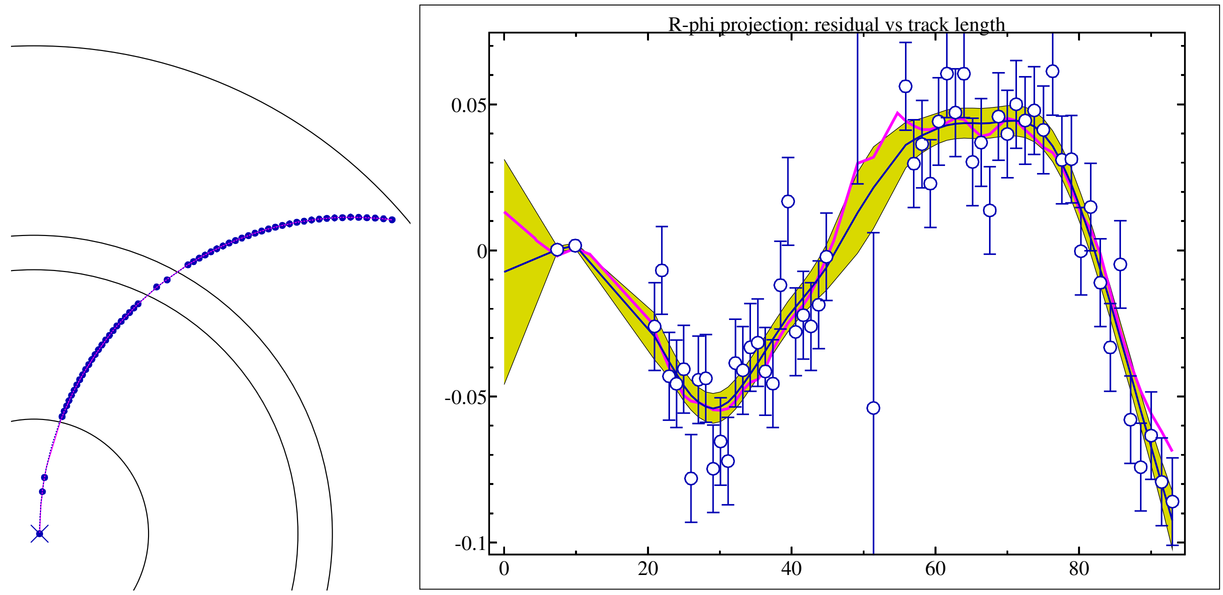

Fig. 3. A 200 MeV/c track in a detector, similar to the H1 detector. On the left the hits in the $r\phi$-plane (perpendicular to the magnetic-field direction) are show together with the result of a circle fit. The residuals of the hits w.r.t. the circle fit are shown on the right as a function of the track length. The lines and the band representing the result of the broken-line fit are explained in section 4.

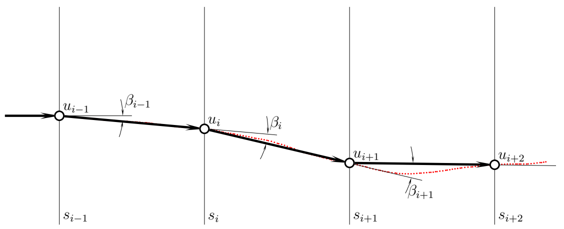

Fig. 4. Particle trajectory with fitted residuals u_i and kink angles β_i.

between the straight line segments. Expectation values $E[\beta_i]$ are zero and the variances $V[\beta_i]$ are the sum of the variances of the angles ψ_{left} and ψ_{right} of the layers between detector planes (eq. (3)):

$$V[\beta_i] = V[\psi_{\text{right},i-1}] + V[\psi_{\text{left},i}] \; .$$

The approximation of the true trajectory is given by the points (s_i, u_i). There are $(n-2)$ kink angles β_i, which are (to good approximation) linear functions of the values u_{i-1}, u_i and u_{i+1},

$$\beta_i = [u_{i-1}\,\delta_{i-1} - u_i\,(\delta_{i-1}+\delta_i) + u_{i+1}\,\delta_i] \qquad (4)$$

with the definition $\delta_i = 1/(s_{i+1} - s_i)$. The angles β_i, which all have an expectation value of zero and a variance given by the multiple scattering theory, can be considered as $(n-2)$ measurements in addition to the n measurements y_i, and these total $(2n-2)$ measurements allow the determination of estimates of the n true points u_i in a linear least squares fit by minimizing the function

$$\boxed{S(\boldsymbol{u}) = \sum_{i=1}^{n} \frac{(y_i - u_i)^2}{\sigma_i^2} + \sum_{i=2}^{n-1} \frac{\beta_i^2}{\sigma_{\beta,i}^2}} \qquad (5)$$

with respect to the values u_i; no explicit parametrization of the trajectory is defined.

The vector $\boldsymbol{u}$ that minimizes the sum-expression $S(\boldsymbol{u})$ is given by the solution of the standard normal equations

$$\boldsymbol{C}_u \boldsymbol{u} = \boldsymbol{r}; \;\; \boldsymbol{C}_u = \begin{pmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{21} & C_{22} & C_{23} & C_{24} & & \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & \\ & C_{42} & C_{43} & C_{44} & C_{45} & \dots \\ & & C_{53} & C_{54} & C_{55} & \dots \\ & & & & & \ddots \end{pmatrix} \qquad (6)$$

of linear least squares. The matrix $\boldsymbol{C}_u$ is a symmetric band matrix, where all elements outside the narrow band (bandwidth $m = 2$) vanish. A fast solution with computing time $\propto n$ is based on the (Cholesky) decomposition of the matrix $\boldsymbol{C}_u$ according to $\boldsymbol{C}_u = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\text{T}}$, where the matrix $\boldsymbol{L}$ is a left unit triangular matrix (diagonal elements are 1) and $\boldsymbol{D}$ is a diagonal matrix; the band structure is kept in this decomposition. The vector $\boldsymbol{u}$ is determined in the steps

$$\begin{aligned} &\text{decompose} & \boldsymbol{C}_u &= \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}} & (6n)\\ &\text{solve} & \boldsymbol{L}\boldsymbol{v} &= \boldsymbol{r}_u & (2n)\\ &\text{solve} & \boldsymbol{L}^{\mathrm{T}}\boldsymbol{u} &= \boldsymbol{D}^{-1}\boldsymbol{v} & (3n)\ . \end{aligned}$$

The number of operations (multiplication, division) per step is indicated in the equations; in total $11n$ operations are needed.

After the reconstruction of the trajectory the corrections Δz_0 and $\Delta(\tan\lambda)$ at track start are calculated from the two first $\boldsymbol{u}$-values u_1 and u_2 from the fitted trajectory and added to the initial approximations. In order to calculate the covariance matrix of the track parameters a few elements of the covariance matrix $\boldsymbol{V}_u \equiv \boldsymbol{C}_u^{-1}$ are required. A special method[4] can be used to calculate those elements of the inverse matrix which are in the band of the original matrix, in a computation time linear in n, using the decomposition $\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}}$; for the bandwidth of $m = 2$ there are only $6n$ operations.

4. Tracking in the $r\phi$-plane

Corrections to the parameters κ, d_{ca} and ϕ_0 are determined in a fit of the residuals in the $r\phi$-plane; in addition to the parameters of section 3 there is a curvature correction $\Delta\kappa$. Corrections Δd_{ca} and $\Delta\phi_0$ are calculated from the first two $\boldsymbol{u}$-values u_1 and u_2. The mean value of the kink angle β_i, as defined in equation (4), is now different from zero, due to the magnetic deflection. The magnetic deflection is taken into account by the *re-definition* of the kink angle

$$\beta_i = [\ldots] \quad + (a_{i-1} + a_i)\cdot \Delta\kappa/2 \qquad (7)$$

(compare equation (4); a_i is the distance between the points i and $i+1$) in the expression of equation (5), with $E[\beta_i] = 0$, and this has to be used in the function $S(\boldsymbol{u}, \Delta\kappa)$ to be minimized, which now depends on the additional parameter $\Delta\kappa$. The solution of the minimization problem is only slightly more complicated. The linear least squares expression $S(\boldsymbol{u}, \Delta\kappa)$ is minimized by the solution of the matrix equation:

$$\left(\begin{array}{c|c} C_\kappa & \boldsymbol{c}^{\mathrm{T}} \\ \hline \boldsymbol{c} & \boldsymbol{C}_u \end{array}\right) \left(\begin{array}{c} \Delta\kappa \\ \hline \boldsymbol{u} \end{array}\right) = \left(\begin{array}{c} \boldsymbol{r}_\kappa \\ \hline \boldsymbol{r}_u \end{array}\right) , \qquad (8)$$

where $\boldsymbol{C}_u$ is as before in section 3, C_κ is a scalar and $\boldsymbol{c}$ is a vector. The solution with the steps

$$\begin{aligned} \boldsymbol{C}_u &= \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}} & (6n)\\ \boldsymbol{C}_u\boldsymbol{z} &= \boldsymbol{c} & (5n)\\ \boldsymbol{B}_\kappa &= \left(\boldsymbol{C}_\kappa - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{z}\right)^{-1} & (n+1)\\ \Delta\kappa &= \boldsymbol{B}_\kappa\left(\boldsymbol{r}_\kappa - \boldsymbol{z}^{\mathrm{T}}\boldsymbol{r}_u\right) & (n+1)\\ \boldsymbol{C}_u\widetilde{\boldsymbol{u}} &= \boldsymbol{r}_u & (5n)\\ \boldsymbol{u} &= \widetilde{\boldsymbol{u}} - \boldsymbol{z}\Delta\kappa & (n) \end{aligned}$$

requires again a number of operations with is proportional to n. The submatrix $\boldsymbol{V}_u$ at the position of the matrix $\boldsymbol{C}_u$ in the inverse matrix is $\boldsymbol{C}_u^{-1} + \boldsymbol{z}B_\kappa\boldsymbol{z}^{\mathrm{T}}$, which again allows the calculation of the covariance matrix of the parameters in a number of operations proportional to n. Figure 3 shows on the right the true simulated and the fitted trajectory, together with the $\pm 1\sigma$ band around the fitted trajectory.

Summary

The proposed algorithm allows fast track fits, fully taking into account multiple scattering; extensions to include energy loss and magnetic-field inhomogeneity are possible. The algorithm gives the full information on every measured point: the fitted value with propagated error, and pulls of position and kink angle. The new algorithm is faster by a factor of six in comparison with the Kalman filter (under test conditions), and gives the result without iterations or recursion.

References

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