Quantum Mechanics
Problem set 1, MT 2008

Some quick questions to check familiarity with the apparatus

1 What physical phenomenon requires us to work with probability amplitudes rather than just with probabilities, as in other fields of endeavour?

2 What properties cause complete sets of amplitudes to constitute the elements of a vector space?

3 V' is the adjoint space of the vector space V. What objects comprise V'?

4 Given that $|\psi\rangle = \mathrm{e}^{\mathrm{i}\pi/5}|a\rangle + \mathrm{e}^{\mathrm{i}\pi/4}|b\rangle$, express $\langle\psi|$ as a linear combination of $\langle a|$ and $\langle b|$.

5 What properties characterise the bra $\langle a|$ that is associated with the ket $|a\rangle$?

6 An electron can be in one of two potential wells that are so close that it can "tunnel" from one to the other. Its state vector can be written

$$|\psi\rangle = a|A\rangle + b|B\rangle,$$

where $|A\rangle$ is the state of being in the first well and $|B\rangle$ is the state of being in the second well and all kets are correctly normalised. What is the probability of finding the particle in the first well given that: (a) $a = \mathrm{i}/2$; (b) $b = \mathrm{e}^{\mathrm{i}\pi}$; (c) $b = \frac{1}{3} + \mathrm{i}/\sqrt{2}$?

7 An electron can "tunnel" between potential wells that form a chain, so its state vector can be written

$$|\psi\rangle = \sum_{-\infty}^{\infty} a_n|n\rangle,$$

where $|n\rangle$ is the state of being in the nth well, where n increases from left to right. Let

$$a_n = \frac{1}{\sqrt{2}}\left(\frac{-\mathrm{i}}{3}\right)^{|n|/2}\mathrm{e}^{\mathrm{i}n\pi}.$$

a. What is the probability of finding the electron in the nth well?
b. What is the probability of finding the electron in well 0 or anywhere to the right of it?

8 How is a wave-function $\psi(x)$ written in Dirac's notation? What's the physical significance of the complex number $\psi(x)$ for given x?

9 Let Q be an operator. Under what circumstances is the complex number $\langle a|Q|b\rangle$ equal to the complex number $(\langle b|Q|a\rangle)^*$ for any state $|a\rangle$ and $|b\rangle$?

10 Let Q be the operator of an observable and let $|\psi\rangle$ be the state of our system.
a. What are the physical interpretations of $\langle\psi|Q|\psi\rangle$ and $|\langle q_n|\psi\rangle|^2$, where $|q_n\rangle$ is the n^{th} eigenket of the observable Q?
b. What is the operator $\sum_n |q_n\rangle\langle q_n|$, where the sum is over all eigenkets of Q?
c. If $u_n(x)$ is the wavefunction of the state $|q_n\rangle$, write dow an integral that evaluates to $\langle q_n|\psi\rangle$.

11 What does it mean to say that two operators commute? What is the significance of two observables having mutually commuting operators?

Given that the commutator $[P, Q] \neq 0$ for some observables P and Q, does it follow that for all $|\psi\rangle \neq 0$ we have $[P, Q]|\psi\rangle \neq 0$?

12 Let $\psi(x, t)$ be the correctly normalised wavefunction of a particle of mass m and potential energy $V(x)$. Write down expressions for the expectation values of (a) x; (b) x^2; (c) the momentum p_x; (d) p_x^2; (e) the energy.

What is the probability that the particle will be found in the interval (x_1, x_2)?

13 Write down the time-independent (TISE) and the time-dependent (TDSE) Schrödinger equations. Is it necessary for the wavefunction of a system to satisfy the TDSE? Is it necessary for the wavefunction of a system to satisfy the TISE?

14 Why is the TDSE first-order in time, rather than second-order like Newton's equations of motion?

Now some problems that require a little calculation

15 A system has a time-independent Hamiltonian that has spectrum $\{E_n\}$. Prove that the probability P_k that a measurement of energy will yield the value E_k is is time-independent.

16 A particle is confined in a potential well such that its allowed energies are $E_n = n^2\mathcal{E}$, where $n = 1, 2, \ldots$ is an integer and $\mathcal{E}$ a positive constant. The corresponding energy eigenstates are $|1\rangle$, $|2\rangle$, $\ldots$, $|n\rangle, \ldots$. At $t = 0$ the particle is in the state

$$|\psi(0)\rangle = 0.2|1\rangle + 0.3|2\rangle + 0.4|3\rangle + 0.843|4\rangle.$$

- **a**. What is the probability, if the energy is measured at $t = 0$ of finding a number smaller than $6\mathcal{E}$?
- **b**. What is the mean value and what is the rms deviation of the energy of the particle in the state $|\psi(0)\rangle$?
- **c**. Calculate the state vector $|\psi\rangle$ at time t. Do the results found in (a) and (b) for time t remain valid for arbitrary time t?
- **d**. When the energy is measured it turns out to be $16\mathcal{E}$. After the measurement, what is the state of the system? What result is obtained if the energy is measured again?

Show that for properly normalised ψ, $\sum_r P(q_r|\psi) = 1$. Why is this significant? Show further that the expectation of Q is $\langle Q\rangle \equiv \int_{-\infty}^{\infty} \psi^* \hat{Q}\psi \, \mathrm{d}x$.

17 Find the energy of neutron, electron and electromagnetic waves of wavelength 0.1 nm.

18 Neutrons are emitted from an atomic pile with a Maxwellian distribution of velocities for temperature 400 K. Find the most probable de Broglie wavelength in the beam.

19 A beam of neutrons with energy E runs horizontally into a crystal. The crystal transmits half the neutrons and deflects the other half vertically upwards. After climbing to height H these neutrons are deflected through 90° onto a horizontal path parallel to the originally transmitted beam. The two horizontal beams now move a distance L down the laboratory, one distance H above the other. After going distance L, the lower beam is deflected vertically upwards and is finally deflected into the path of the upper beam such that the two beams are co-spatial as they enter the detector. Show that the wavenumbers k, k' of the lower and upper beams are related by

$$k' \simeq k\left(1 - \frac{m_{\mathrm{n}}gH}{2E}\right).$$

In an actual experiment (R. Colella et al., 1975, Phys. Rev. Let., 34, 1472) $E = 0.042\,\mathrm{eV}$ and $LH \sim 10^{-3}\,\mathrm{m}^2$ (the actual geometry was slightly different). Determine the phase difference between the two beams at the detector. Sketch the intensity in the detector as a function of H.

20 A particle moves in the potential $V(\mathbf{x})$ and is known to have energy E_n. (a) Can it have well defined momentum for some particular $V(\mathbf{x})$? (b) Can the particle simultaneously have well-defined energy and position?

21 The states $\{|1\rangle, |2\rangle\}$ form a complete orthonormal set of states for a two-state system. With respect to these basis states the operator σ_y has matrix

$$\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Could σ be an observable? What are its eigenvalues and eigenvectors in the $\{|1\rangle, |2\rangle\}$ basis? Determine the result of operating with σ_y on the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle).$$

22 A three-state system has a complete orthonormal set of states $|1\rangle, |2\rangle, |3\rangle$. With respect to this basis the operators H and B have matrices

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω and b are real constants.

a. Are H and B Hermitian?
b. Do the eigenvectors of either H or B define a unique basis of eigenvectors?
c. Show that H and B commute. Give a basis of eigenvectors common to H and B.

23 Given that A and B are Hermitian operators, show that $\mathrm{i}[A, B]$ is a Hermitian operator.

24 Given a ordinary function $f(x)$ and an operator R, the operator $f(R)$ is defined to be

$$f(R) = \sum_i f(r_i)|r_i\rangle\langle r_i|,$$

where r_i are the eigenvalues of R and $|r_i\rangle$ are the associated eigenkets. Show that when $f(x) = x^2$ this definition implies that $f(R) = RR$, that is, that operating with $f(R)$ is equivalent to applying the operator R twice.

25 Show that if there is a complete set of mutual eigenkets of the Hermitian operators A and B, then $[A, B] = 0$.

26 Given that $(AB)^\dagger = B^\dagger A^\dagger$, show that

$$(ABCD)^\dagger = D^\dagger C^\dagger B^\dagger A^\dagger.$$

27 Prove that

$$[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC.$$

Explain the similarity with the rule for differentiating a product.

28 Show that a classical harmonic oscillator satisfies the virial equation $2\langle \mathrm{KE}\rangle = \alpha\langle \mathrm{PE}\rangle$ and determine the relevant value of α.

29 A classical fluid of density $\rho(\mathbf{x})$ flows with velocity $\mathbf{v}(\mathbf{x})$. By differentiating with respect to time the mass $m \equiv \int_V \mathrm{d}^3\mathbf{x}\,\rho$ contained in an arbitrary volume V, show that conservation of mass requires that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

Hint: the flux of matter at any point is $\rho\mathbf{v}$ and the integral of this flux over the boundary of V must equal the rate of accumulation of mass within V.

$\mathbf{J}$ is defined to be

$$\mathbf{J}(\mathbf{x}) \equiv \frac{\mathrm{i}\hbar}{2m}\Big(\psi\nabla\psi^* - \psi^*\nabla\psi\Big),$$

where $\psi(\mathbf{x})$ is the wavefunction of a spinless particle of mass m. Working from the TDSE, show that

$$\frac{\partial|\psi|^2}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Give a physical interpretation of this result.

Show that when we write the wavefunction in amplitude-modulus form, $\psi = |\psi|\mathrm{e}^{\mathrm{i}\theta}$,

$$\mathbf{J} = |\psi|^2\frac{\hbar\nabla\theta}{m}.$$

Interpret this result physically. Given that $\psi = A\mathrm{e}^{\mathrm{i}(kz-\omega t)} + B\mathrm{e}^{-\mathrm{i}(kz+\omega t)}$, where A and B are constants, show that

$$\mathbf{J} = v\big(|A|^2 - |B|^2\big)\,\hat{\mathbf{z}},$$

where $v = \hbar k/m$. Interpret the result physically.

30 A system AB consists of two non-interacting parts A and B. The dynamical state of A is described by $|a\rangle$, and that of B by $|b\rangle$, so $|a\rangle$ satisfies the TDSE for A and similarly for $|b\rangle$. What is the ket describing the dynamical state of AB? In terms of the Hamiltonians H_A and H_B of the subsystems, write down the TDSE for the evolution of this ket and show that it is automatically satisfied.

Do H_A and H_B commute? How is the TDSE changed when the subsystems are coupled by a small dynamical interaction $H_{\rm int}$? If A and B are harmonic oscillators, write down H_A, H_B. The oscillating particles are connected by a weak spring. Write down the appropriate form of the interaction Hamiltonian $H_{\rm int}$. Does H_A commute with $H_{\rm int}$?

31 Consider a system of two particles of mass m that each move in one dimension along a given rod. Let $|x\rangle_1$ be the state of the first particle when it's at x and $|y\rangle_2$ be the state of the second particle when it's at y. A complete set of states of the pair of particles is $\{|xy\rangle\} = \{|x\rangle_1|y\rangle_2\}$. Write down the Hamiltonian of this system given that the particles attract one another with a force that's equal to C times their separation.

Suppose the particles experience an additional potential

$$V(x, y) = \tfrac{1}{2}C(x + y)^2.$$

Show that the dynamics of the two particles is now identical with the dynamics of a single particle that moves in two dimensions in a particular potential $\Phi(x, y)$, and give the form of Φ.