**What is a linear vector space? Give some examples to illustrate the breadth of this definition.**

To form a vector space we require a set of objects V, in which we might denote an element as $|a\rangle$ if using Dirac notation, and two operations. Specifically:

Addition, we can add two objects in V together in a way that is both commutative and associative:

$$|a\rangle + |b\rangle = |b\rangle + |a\rangle \quad \text{commutative,}$$
$$(|a\rangle + |b\rangle) + |c\rangle = |a\rangle + (|b\rangle + |c\rangle) \quad \text{associative,}$$

thus the order or grouping of addition does not matter.

Multiplication, we can multiply any object in V by a complex scalar $z$ in a way that is commutative, associative and distributive, namely:

$$w \cdot z \cdot |a\rangle = z \cdot w \cdot |a\rangle \quad \text{commutative,}$$
$$w \cdot (z \cdot |a\rangle) = (w \cdot z) \cdot |a\rangle \quad \text{associative,}$$
$$(z+w)|a\rangle = z|a\rangle + w|a\rangle \quad \text{distributive,}$$
$$z(|a\rangle + |b\rangle) = z|a\rangle + z|b\rangle \quad \text{distributive,}$$

These properties ensure that objects in V behave in a similar way to arithmetic operations with scalars.

We can say that V forms a linear vector space if it is additionally:

(i) Closed under addition, so $|a\rangle + |b\rangle = |c\rangle$ gives $|c\rangle \in V$ for any $|a\rangle, |b\rangle \in V$.

(ii) closed under multiplication, so $z \in \mathbb{C}$ and $|a\rangle \in V$ gives $|c\rangle = z \cdot |a\rangle$ with $|c\rangle \in V$ as well.

(iii) There exists a null element, which we might denote as $|0\rangle$, such that $|a\rangle + |0\rangle = |a\rangle$, for all $|a\rangle \in V$.

(iv) There exists, for any $|a\rangle \in V$, another element $|a'\rangle \in V$ obeying $|a\rangle + |a'\rangle = |0\rangle$.

By obeying these properties the elements $|a\rangle \in V$ are called "vectors" and V is a vector space.

Examples include, matrices and classes of matrices, complex numbers (1D complex vector space or 2D real), and even sets of continuous functions such as polynomials, sines/cosines.