Mathematical Methods (Second Year) MT 2006

Problem Set 5: Fourier Analysis

A. Fourier Series

1. Sketch graphs of the following functions in the range $-2\pi < x < 2\pi$ given that all the functions are periodic with period 2π:

   (a) $f(x) = |x|, \quad -\pi < x < \pi;$

   (b) $f(x) = x, \quad -\pi < x < \pi;$

   (c) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

2. State whether the following functions are even, odd or neither even nor odd:

   (a) $\sin(x)$ (b) $\cos(x)$ (c) $\sin^2(x)$ (d) $\cos^2(x)$ (e) $\sin(2x)$ (f) $\cos(2x)$

   (g) $x\cos(x)$ (h) $\sin(x) + \cos(x)$ (i) $x + \sin(x)$ (j) e^x

3. Find the Fourier series for the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}. \tag{1}$$

Fourier Sine and Cosine Series

Non-periodic functions defined on an interval $[a, b]$ can be represented as Fourier series by *continuing* them outside $[a, b]$ in order to make them periodic. An example of what we mean by this is shown in Figs 1 and 2. In Fig. 2 two different periodic continuations of the function $f(x)$ defined on the interval $[0, a]$ are shown.

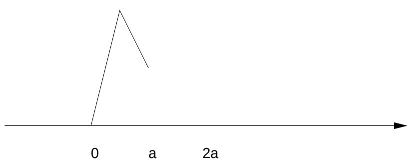


Figure 1: A function $f(x)$ defined on the interval $[0, a]$.

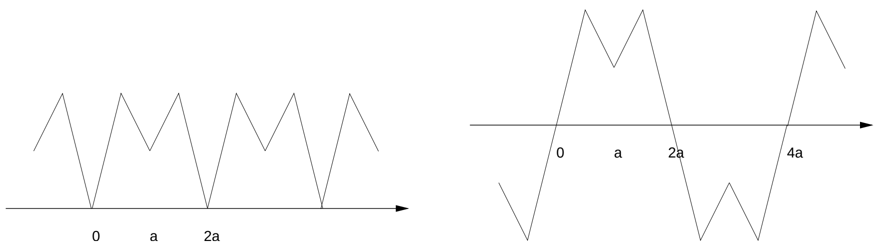


Figure 2: Two different periodic continuations of $f(x)$ with periods $2a$ and $4a$ respectively.

Both continuations coincide with $f(x)$ on the interval $[0, a]$ and both continuations can be expressed in terms of Fourier series. These Fourier series will be different, but both of them will coincide with $f(x)$ on the interval $[0, a]$! In the simplest case one has a function $g(x)$ defined on $[0, L]$ with $g(0) = g(L)$. Then there are two generic ways of continuing this function to the interval: a symmetric continuation $g_1(x) = g_1(-x) = g(x)$ and and antisymmetric one $g_2(x) = -g_2(-x) = g(x)$. The Fourier series for $g_1(x)$ will only contain cosines and is called the *Fourier cosine series*. The Fourier series for $g_2(x)$ will only contain sines and is called the *Fourier sine series*.

4. Find (a) the Fourier sine series and (b) the Fourier cosine series for the function

$$f(x) = x \sin x, \qquad 0 < x < \pi.$$

In order to determine the Fourier sine series, you need to extend the function to the interval $[-\pi, 0]$ by requiring that $f(-x) = -f(x)$. Now determine the Fourier sine series for the resulting odd function on the interval $[-\pi, \pi]$. This Fourier series involves only sines as the extended function is odd. The Fourier sine series equals $f(x)$ on the original interval $[0, \pi]$.

5. Find (a) the Fourier sine series and (b) the Fourier cosine series for the function $f(x) = x^2$ in the interval $0 < x < 2$.

B. Fourier Transforms and Generalized Functions

Fourier transforms   [Def: $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$]

6. Evaluate $\tilde{f}(k)$ for

$$\begin{array}{rcl} f(x) &=& (a-x)/a^2 \quad \text{for} \quad 0 \le x \le a \\ f(x) &=& (a+x)/a^2 \quad \text{for} \quad -a \le x \le 0 \\ f(x) &=& 0 \quad \text{for} \quad x \le -a, \; x \ge a \end{array}$$

(you may find it helpful to sketch $f(x)$).

Sketch $\tilde{f}(k)$ as a function of k, marking in particular (a) the value of $\tilde{f}(k)$ at $k = 0$ (b) the position of the first zeros on either side of $k = 0$.

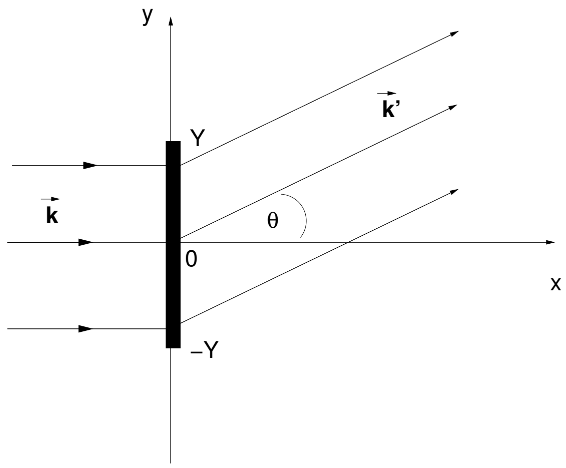
Explain the reason for the value of $\tilde{f}(0)$ in geometrical terms.

7.* N.B. This problem is for students who have already taken the complex analysis short option (and they are strongly advised to do it).

Evaluate the Fourier transforms of the following functions for $a > 0$ (use the reside theorem where appropriate)

(a) $f(x) = \frac{2a}{a^2+x^2}$,   (b) $f(x) = \exp(-a|x|)$,   (c) $f(x) = \frac{1}{\cosh(x)}$.

8.* A direct application of Fourier transform techniques is *Fraunhofer diffraction*. See the discussion on p. 443 of Riley/Hobson/Bence. Consider a beam of monochromatic light of wavelength λ incident on a two dimensional screen of width $2Y$. The direction of the incident beam is



specified by the wave vector $\vec{k}$. The magnitude of $\vec{k}$ is given by the wave number $k = |\vec{k}| = \frac{2\pi}{\lambda}$. The essential quantity in the Fraunhofer diffraction pattern is the dependence of the observed amplitude (and hence intensity) on the angle θ between the viewing direction (which we specify through a vector $\vec{k}'$) and the direction of the incident beam $\vec{k}$. We suppose that at the position

$(0, y)$ the amplitude of the transmitted light is $f(y)$ per unit length in the y-direction (note that $f(y)$ may be complex). The function $f(y)$ is called an *aperture function*. Both the screen and the beam are assumed infinite in the z-direction. The total light amplitude at a position

$$\vec{r}_0 = x_0\ \vec{e}_x + y_0\ \vec{e}_y\ , \tag{2}$$

with $x_0 > 0$ will be the superposition of all Huygens' wavelets originating from the various parts of the screen. For very large $|\vec{r}_0$ these can be treated as plane waves to give

$$A(\vec{r}_0) = \int_{-Y}^{Y} dy\ f(y) \frac{\exp\Big(i\vec{k}' \cdot (\vec{r}_0 - y\vec{e}_y)\Big)}{|\vec{r}_0 - y\vec{e}_y|}. \tag{3}$$

Here the factor $\exp\Big(i\vec{k}' \cdot (\vec{r}_0 - y\vec{e}_y)\Big)$ represents the phase change undergone bby the light in travelling from the point $y\vec{e}_y$ on the screen to the point $\vec{r}_0$ and the denominator represents the reduction in amplitude with distance. Writing $\vec{k}' = k\cos(\theta)\vec{e}_x + k\sin(\theta)\vec{e}_y$ and assuming that $|\vec{r}_0| \gg Y$ we can approximate $|\vec{r}_0 - y\vec{e}_y| \approx |\vec{r}_0|$ and hence approximate (3) as

$$A(\vec{r}_0) \approx \frac{\exp i\vec{k}' \cdot \vec{r}_0}{|\vec{r}_0|} \int_{-Y}^{Y} dy\ f(y) \exp\Big(-iky\sin(\theta)\Big) \tag{4}$$

Now using that $f(y) = 0$ for $|y| > Y$ we can extende the integration limits to $\pm\infty$. Then the intensity in direction θ is given by

$$I(\theta) = |A|^2 = \frac{2\pi}{|\vec{r}_0|^2} |\tilde{f}(k\sin(\theta))|^2 \equiv \frac{2\pi}{|\vec{r}_0|^2} \widehat{I}(\theta). \tag{5}$$

Use this to calculate $\widehat{I}(\theta)$ for

(a) The "single finite slit" diffraction pattern, given by the aperture function

$$\begin{aligned} f(x) &= 1 \quad \text{for} \quad |x| \le b/2 \\ &= 0 \quad \text{otherwise;} \end{aligned}$$

(b) The "two narrow slits" diffraction pattern, given by the aperture function

$$f(x) = \delta(x + a/2) + \delta(x - a/2);$$

(c) The "two finite slits" diffraction pattern, given by

$$\begin{aligned} f(x) &= 1 \quad \text{for} \quad \tfrac{a-b}{2} \le x \le \tfrac{a+b}{2} \\ &= 1 \quad \text{for} \quad \tfrac{(-a-b)}{2} \le x \le \tfrac{-a+b}{2} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $b < a$. How is the result (c) related to those for (a) and (b)? Can you explain this in terms of the convolution theorem?

9. By taking the Fourier transform of the differential equation

$$\frac{d^2\Phi}{dx^2} - K^2\Phi = f(x)\ , \tag{6}$$

show that its solution can be written as

$$\Phi(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\ e^{ikx}\ \frac{\tilde{f}(k)}{k^2 + K^2}. \tag{7}$$

10. Show that

    (a) The Fourier transform of $f(ax)$ is $\frac{1}{a}\tilde{f}(k/a)$;

    (b) The Fourier transform of $f(a+x)$ is $e^{ika}\tilde{f}(k)$;

    (c) The Fourier transform of $e^{iqx}f(x)$ is $\tilde{f}(k-q)$;

    (d) The Fourier transform of $\frac{df(x)}{dx}$ is $ika\tilde{f}(k)$;

    (e) The Fourier transform of $xf(x)$ is $i\frac{d\tilde{f}(k)}{dk}$.

11. *Use of Fourier series in boundary value problems*

    The potential $V(x, y)$ satisfies $\frac{\partial^2 V}{dx^2} + \frac{\partial^2 V}{\partial y^2} = 0$ and the boundary conditions

    (i)$V = 0$ on $y = 0$ and $y = a$;

    (ii) $V \to 0$ as $x \to \infty$;

    (iii) $V = V_0$ on $x = 0$ for $0 < y < a$.

    Show that the potential in the region $0 < y < a, x > 0$ is given by

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,\ldots} \frac{1}{n} \sin\left(\frac{n\pi y}{a}\right) e^{-n\pi x/a}.$$