Problem sheet - MT tutorial 6

Main questions

Q1. Eigenfunctions and eigenvalues

The eigenvalue equation for the SHO is

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right)\phi = E\phi$$

where ω is the classical frequency of the oscillator.

i) Show that by making the change of variables $x = y\sqrt{\frac{\hbar}{m\omega}}$ (42) becomes

$$\frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right)\phi = E\phi \tag{43}$$

ii) Show that $\phi(y) = e^{-\frac{y^2}{2}}$ satisfies the eigenvalue equation for the SHO and find E. In fact this is the ground state. Show that the correctly normalized wavefunction is

$$\phi_0(x) = \left(\frac{1}{\pi a^2}\right)^{\frac{1}{4}} \exp(-x^2/2a^2) \quad \text{where} \quad a^2 = \hbar/m\omega. \tag{44}$$

iii) Find the expectation values of x, x^2, p and p^2 for a particle in the ground state. [Hint: for $\langle p^2\rangle$ use $p^2/2m + \frac{1}{2}m\omega^2 x^2 = E$].

iv) Defining $\Delta x = [\langle(x - \langle x\rangle)^2\rangle]^{\frac{1}{2}}$, $\Delta p = [\langle(p - \langle p\rangle)^2\rangle]^{1/2}$ show that in this case $\Delta x\Delta p = \frac{1}{2}\hbar$. Comment on this result.

You will need the integrals

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}; \qquad \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}.$$

[JMR QM Q6.1]

Q2. Some pictures

On the same diagram, plot carefully (paying particular attention to the points of intersection of the various curves)

i) $\phi_0(x)$ from 44 versus x, indicating where $\frac{d^2\phi_0}{dx^2} = 0$

ii) the potential energy $\frac{1}{2}m\omega^2 x^2$ versus x

iii) the total energy $E = \frac{1}{2}\hbar\omega$ versus x

iv) the region in x to which the particle would be confined according to classical mechanics.

Why do the x-values such that $\frac{d^2\phi_0}{dx^2} = 0$ lie at the limits of the classically allowed region?

[JMR QM Q6.2]

Q3. Determining the spectrum

The form (43) strongly suggests that we should try factorizing the differential operator. So define

$$
\begin{aligned}
D^\dagger &= \frac{1}{\sqrt{2}}\left(-\frac{d}{dy}+y\right) \\
D &= \frac{1}{\sqrt{2}}\left(\frac{d}{dy}+y\right)
\end{aligned}
$$

i) Show that

$$
\begin{aligned}
D^\dagger D f &= \frac{1}{2}\left(-\frac{d^2 f}{dy}+y^2 f-f\right) \\
D\, D^\dagger f &= \frac{1}{2}\left(-\frac{d^2 f}{dy}+y^2 f+f\right)
\end{aligned}
$$

and hence that (43) may be written

$$\hbar\omega\left(D^\dagger D+\frac{1}{2}\right)\phi = E\phi \tag{48}$$

and that

$$D^\dagger D - D\, D^\dagger = -1$$

ii) Now *assume* that ϕ satisfies (48). Show that $\phi' = D\,\phi$ also satisfies (48) but with E replaced by $E' = E - \hbar\omega$. (Most easily done by acting on (48) with D and then using the commutator to reverse the order of D and $D^\dagger$.)

iii) Explain why there must be a ϕ_0 satisfying $D\,\phi_0 = 0$. Writing this condition out explicitly gives a first order differential equation for ϕ_0; solve it. To what value of E does ϕ_0 correspond?

iv) Now show that if ϕ satisfies (48) then $\phi' = D^\dagger\phi$ also satisfies (48) but with E replaced by $E' = E + \hbar\omega$.

v) Now assemble everything to give the spectrum E_n and a recipe for generating the eigenfunctions ϕ_n. Write out the first three eigenfunctions explicitly and plot them on a graph.

vi) Is the ground state wavefunction an even or odd function of x? How do the excited states behave under $x \to -x$? (This property is called the *parity* of the state.)

[JMR QM Q6.3]

Q4. The slicker operator approach

Of course what we did in the previous question didn't really depend on a differential equation; just on the properties of some operators. So actually it can be done at a more abstract level without reference to differential operators at all. So let

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

$$A = \sqrt{\frac{m\omega}{2}}x + i\frac{p}{\sqrt{2m\omega}}$$

$$A^\dagger = \sqrt{\frac{m\omega}{2}}x - i\frac{p}{\sqrt{2m\omega}}$$

i) Show that $[A, A^\dagger] = \hbar$ and $H = \omega A^\dagger A + \hbar\omega/2$.

ii) Show that $[H, A] = -\hbar\omega A$ and $[H, A^\dagger] = \hbar\omega A^\dagger$

iii) Assume that $H|\psi\rangle = E|\psi\rangle$; show that $|\psi'\rangle = A|\psi\rangle$ satisfies $H|\psi'\rangle = (E - \hbar\omega)|\psi'\rangle$. Deduce that there must be a state $|0\rangle$ satisfying $A|0\rangle = 0$ and give its energy.

iv) Show that $|\psi'\rangle = A^\dagger|\psi\rangle$ satisfies $H|\psi'\rangle = (E + \hbar\omega)|\psi'\rangle$. Now you can deduce the spectrum E_n and how the corresponding states $|n\rangle$ are related to $|0\rangle$.

v) It's easy to compute the correct normalization too. Being careful we have

$$|n+1\rangle = C_n A^\dagger|n\rangle$$

where the states are all normalised and C_n is a constant. Show that

$$1 = \langle n+1|n+1\rangle = |C_n|^2\hbar(n+1).$$

This tells you C_n; find the constant N_n such that

$$|n\rangle = N_n(A^\dagger)^n|0\rangle$$

is correctly normalized.

[JMR QM Q6.4]

Q5. Functions of the annihilation operator on the ground state

Prove that

$$A f(A^\dagger)\,|0\rangle = \frac{df(A^\dagger)}{dA^\dagger}\,|0\rangle\,,$$

where $f(\cdot)$ is some regular function possessing a power series expansion.

Q6. Special state of a harmonic oscillator - the coherent state

A state, denoted as $|\alpha\rangle$ where $\alpha \in \mathbb{C}$, that obeys the equation $A|\alpha\rangle = \alpha|\alpha\rangle$ is called a coherent state. (i) Show that the state $|\alpha\rangle = C\exp(\alpha A^\dagger)|0\rangle$, where C is a normalization constant. (ii) Use the result from Q5 to obtain C. (iii) Expand the state $|\alpha\rangle$ in a series expansion of eigenstates of the number operator $N = A^\dagger A$, called number states $|n\rangle$, and use this to find the probability that the coherent state contains n quanta. What distribution do you obtain? (iv) Compute $\langle\alpha|N|\alpha\rangle$, the average number of quanta in a coherent state.

Q7. Another identity for operators acting on the ground state

Use the result from Q5 to prove that

$$\exp(\lambda A^\dagger) f(A^\dagger)\,|0\rangle = f(A^\dagger + \lambda)\,|0\rangle\,,$$

where $f(\cdot)$ is again a regular function possessing a power series expansion.

Q8. Elevation to a full operator identity

An operator identity is an equation which holds when the operators act on any arbitrary state. Explain why an arbitrary state of a harmonic oscillator can be expressed as $|\psi\rangle = g(A^\dagger)|0\rangle$, where $g(\cdot)$ is some regular function? Use this and the result from Q7 to prove the operator identity.

$$\exp(\lambda A^\dagger) f(A^\dagger) \exp(-\lambda A^\dagger) = f(A^\dagger + \lambda),$$