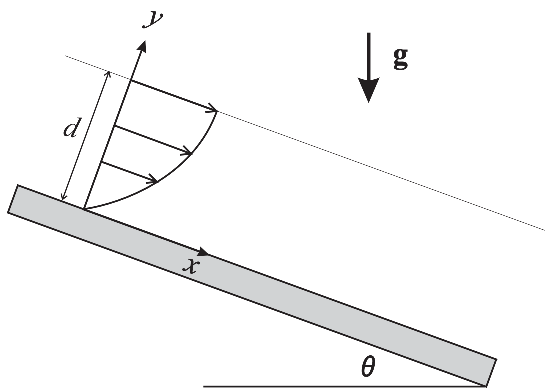
Flows, Fluctuations & Complexity

Problem Set: DRAFT 0

1 Steady flows

4. Consider steady, 2D incompressible viscous flow down an inclined plane, under gravity. Choose axes as shown in the diagram, and assume that the velocity $\mathbf{u}$ depends only on y.



What are the boundary conditions for u and v at $y = 0$? Using incompressibility, show that $v = 0$ throughout the flow. Write down the x- and y-components of the Navier-Stokes equation, and from the y-component show that the pressure is of the form $p(x, y) = -\rho g y \cos\theta + F(x)$, where F is an arbitrary function. Assuming that the pressure at the free surface $y = d$ equals the constant atmospheric pressure p_0, show further that

$$p = p_0 + \rho g(d - y)\cos\theta \ .$$

From the x-component of the N-S equation, together with an appropriate boundary condition at $y = 0$ and the zero tangential stress condition $\mu du/dy = 0$ at the free surface $y = d$, show that

$$u = \frac{g}{2\nu} y(2d - y)\sin\theta \ .$$

Show that the volume flux per unit distance in the z-direction is $gd^3 \sin\theta/(3\nu)$.

2 Waves and radiation

3. Discuss the similarities and differences between the equations governing very viscous fluid flow and the flow of photons through an absorbing medium.

The change in radiance (energy flux per unit area per unit solid angle per unit spectral interval) for radiation at at a specific wavenumber ν flowing through a thin, homogenous slab (thickness δz) of an absorbing and emitting medium is given by

$$\delta L_\nu = k_\nu \rho_a [B_\nu - L_\nu] \delta z \ ,$$

where ρ_a is the density of the medium, k_ν is the mass absorption coefficient and B_ν is the Plank function at the temperature of the medium. With reference to the Beer-Lambert and Kirkhoff laws, explain the physical origin of the terms on the RHS.

Defining *transmittance* between z_1 and z_2 as $\mathcal{T}_\nu(z_1, z_2) = L_\nu(z_2)/L_\nu(z_1)$ and *optical thickness* $\chi_\nu(z_1, z_2) = -\ln[\mathcal{T}_\nu(z_1, z_2)]$, derive Schwarzschild's equation for the rate of change of radiance due to both absorption and emission in a continuous medium, with optical thickness defined relative to the observer receiving the radiance (so L is oriented "down-χ"):

$$-\frac{dL_\nu(z)}{d\chi_\nu(z)} + L_\nu(z) = B_\nu(z) \ .$$

Using the integrating factor $e^{-\chi}$, or otherwise, show that the radiance received by a detector at $\chi = 0$ observing a source emitting plane-parallel radiation with intensity L_s through an inhomogenous medium of optical thickness χ_s is given by

$$L_0 = L_\mathrm{s} e^{-\chi_\mathrm{s}} + \pi \int_0^{\chi_\mathrm{s}} B(\chi) e^{-\chi} d\chi$$

where $B(\chi)$ is the Planck function at optical thickness χ measured from the detector.

Hence show that, if the source radiates as a black body,

$$L_0 = B(0) + \int_0^{\chi_\mathrm{s}} \frac{dB(\chi)}{d\chi} e^{-\chi} \, d\chi$$

and

$$L_0 = B(\chi_s) - \int_0^{\chi_\mathrm{s}} \frac{dB(\chi)}{d\chi} \left(1 - e^{-\chi}\right) d\chi \ .$$

The quantity $1 - e^{-\chi} = 1 - \mathcal{T}_\chi = \mathcal{A}_\chi$ is called the *absorptance* over a given optical path. Explain the physical significance of the various terms in these two expressions in the strong absorption ($\chi_\mathrm{s} \gg 1$) and weak absorption ($\chi_\mathrm{s} \ll 1$) limits.

3 Flows in state-space (dynamical systems)

1. Explain the meaning of the terms *fixed point*, *stable node*, *unstable node*, *saddle point* and *stable manifold* in the context of a phase-space description of a physical system.

A two-mode laser produces photons at two different wavelengths in numbers n_1 and n_2 at the following rates:

$$\frac{dn_1}{dt} = G_1 N n_1 - k_1 n_1\,,$$
$$\frac{dn_2}{dt} = G_2 N n_2 - k_2 n_2\,,$$

where $N = N_0 - \alpha_1 n_1 - \alpha_2 n_2$ is the number of excited atoms and the parameters G_1, G_2, k_1, k_2, α_1, α_2 and N_0 are all positive constants.

a) Find the fixed points of the system, and discuss the special case when $k_1/G_1 = k_2/G_2$.

b) Write down the Jacobian for this system and use it to evaluate the stability of these fixed points.

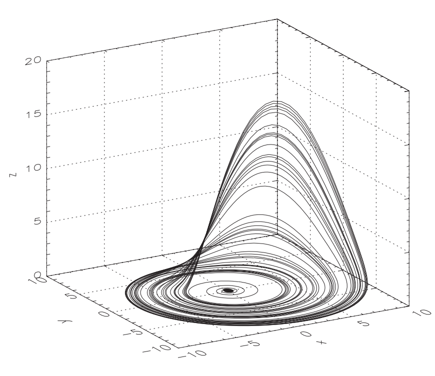
c) Sketch phase portraits for the cases $k_1/G_1 > k_2/G_2$, $k_1/G_1 = k_2/G_2$ and $k_1/G_1 < k_2/G_2$.

4. The Rössler attractor is generated by the dynamical system:

$$\dot{x} = -y - z,$$
$$\dot{y} = x + ay,$$
$$\dot{z} = b + z(x - c),$$

where a, b and c are positive constants. Find the two fixed points of the system.

Write down the Jacobian for the Rössler system. Use it to show that perturbations in the y-direction always grow regardless of location and find expressions for the growth rate of perturbations orthogonal to the y-direction in terms of x and z.



The figure shows an integration of the Rössler system with $a = b = 0.2$ and $c = 5$, with the heavy dot marking the initial conditions close to the more stable fixed point. In this vicinity, show that perturbations in one direction decay rapidly, confining the attractor close to a two-dimensional surface. Discuss the evolution of the trajectory as it moves further away from the fixed point, using your expressions for the growth rate of perturbations to explain how trajectories diverge.

4 Stochastic systems

5. The charge Q on the capacitor in an LRC circuit driven by a stochastic, Gaussian distributed, input voltage $\epsilon(t)$ is given by:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = \epsilon(t)$$

The charge Q is sampled at intervals of time separated by δt.

For small δt, show that this is equivalent to a second-order auto-regressive, or AR(2), process

$$u_t = a_0 z_t + a_1 u_{t-1} + a_2 u_{t-2},$$

where z_t is a unit-variance white noise and

$$\begin{aligned} \frac{R}{L} &= \frac{-a_1 - 2a_2}{a_2 \delta t}, \\ \frac{1}{LC} &= -\frac{1 - a_1 - a_2}{a_2 (\delta t)^2} \end{aligned}$$

By considering lag-0, lag-1, and lag-2 covariances, show that the variance of an AR(2) process is given by

$$\langle u_t^2 \rangle = \frac{(1 - a_2){a_0}^2}{(1 + a_2)(1 - a_1 - a_2)(1 + a_1 - a_2)}$$

Discuss the physical significance of the first two terms in the denominator on the right hand side, paying particular attention to the limit $\delta t \to 0$.

5 Biological systems

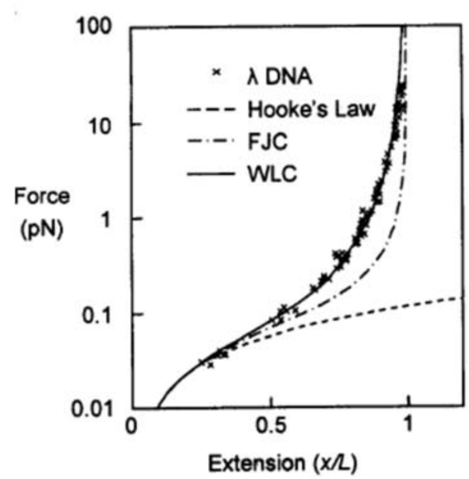
1. Go to www.pdb.org (the Protein Data Bank) and explore links including 'Getting Started' and (from there) 'Looking at Structures'. Inspect the following structures (just enter the id in the search box near the top of the page). Try to look at them with an interactive viewer - see the options given under the thumbnail picture of the molecule on the RHS of the structure page. Alternative viewers (available on the web, and see menu) include Rasmol, the Swiss-PDB Viewer and Pymol - these make it easier to change the way the data is displayed (Pymol was used to make the images used in the lectures). If nothing else works, the 'all images' option will give different static views.

a) 1rcp What is the principal secondary structure motif? What is the function of this protein?

b) 1tnf What is the principal secondary structure motif?

c) 1ytb What is it binding?

Have a look at (left hand menu) Education - Molecule of the Month. Browse the archive, and use the links to look at some of the structures with an interactive viewer. (There is an alternative database which recognises these identifiers:
http://www.ncbi.nlm.nih.gov/entrez/query.fcgi?db=Structure - it has yet another downloadable viewer Cn3D.)



2. The figure above shows a force-extension curve for λ-phage double-stranded DNA of length L ($\sim 10^{-6}$ m). The curve labelled FJC is predicted by the freely jointed chain model of polymer elasticity. What is the functional relationship between force f and extension x for small extensions? Discuss the physical origin of the observed behaviour in the low- and high-force regimes, explaining why the experimental data deviates from the FJC prediction at high applied forces. Copy the FJC curve and, on the same log-normal axes, sketch a second curve predicted by the FJC model for a lower temperature (you need not specify the temperature).

A comment on notation. The Lorentz transformation is written either $\mathcal{L}$ or Λ. 3-vectors are indicated by bold font: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots \mathbf{A}, \mathbf{B}, \mathbf{C} \ldots$; 4-vectors are indicated by capitol letters in this font: $\mathsf{A}, \mathsf{B}, \mathsf{C}, \ldots$; 2nd rank tensors by this font $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$. In the later sections where index notation is used, the font no longer matters but is mostly adhered to anyway. Then Greek indices run over 0,1,2,3 (t, x, y, z); Roman indices run over 1,2,3 (x, y, z). The Minkowski metric is taken as $g = \mathrm{diag}(1, -1, -1, -1)$ (in rectangular coordinates). The scalar product is

$$\mathsf{A} \cdot \mathsf{B} \equiv \mathsf{A}^T g \mathsf{B} \equiv \mathsf{A}^\alpha \mathsf{B}_\alpha$$

In the lectures we have adopted the notation $\Box$ for the 4-gradient operator, by analogy with the familiar $\boldsymbol{\nabla}$. In textbooks there is often no notation offered for a 4-vector gradient operator, because index notation is adopted and one uses ∂^μ. The symbol $\Box$ is then often used for the d'Alembertian operator, which in our notation is $\Box^2$. The summary is

thing	notation here	notation elsewhere
$\left(\frac{1}{c}\left(\frac{\partial}{\partial t}\right), -\boldsymbol{\nabla}\right)$	$\Box$	∂^μ
$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$	$\Box^2$	$\Box$

I believe the use of $\Box$ is less confusing than ∂^μ for learning purposes. We shall eventually go over to ∂^μ when we need to.

Relativity: problems

Andrew M. Steane

1 Basic ideas, simple kinematics and dynamics

(**Lectures 1-3**)

10. Derive the Doppler effect from the invariant $\mathsf{K} \cdot \mathsf{U}$ where K is the 4-wave vector and U is the 4-velocity of the source.

Excited ions in a fast beam have a uniform velocity and emit light on a given internal transition. The wavelength observed in the direction parallel to the beam is 359.5 nm, the wavelength observed in the direction perpendicular to the beam in the laboratory is 474.4 nm. Find the wavelength in the rest frame of the ions, and the speed of the ions in the laboratory. [*Ans.* 406.3 nm, 0.2422c]

15. Show that the motion of a particle in a uniform magnetic field is in general helical, with the period for a cycle independent of the initial direction of the velocity. [Hint: what can you learn from $\mathbf{f} \cdot \mathbf{v}$?]

2 Energy-momentum conservation; collisions; 4-gradient

(**Lectures 4-7**)

8 (i) Pion formation can be achieved by the process $\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \mathrm{p} + \pi_0$. A proton beam strikes a target containing stationary protons. Calculate the minimum kinetic energy which must be supplied to an incident proton to allow pions to be formed, and compare this to the rest energy of a pion.
(ii) An electron collides with another electron at rest to produce a pair of muons by the process $e + e \rightarrow e + e + \mu^+ + \mu^-$. Show that the threshold momentum of the incident electron for this process is

$$p_{\rm th} = 2Mc(1 + M/m)\sqrt{1 + 2m/M}$$

where m, M are the masses of the electron, muon, respectively.
(iii) A photon is incident on a stationary proton. Find, in terms of the rest masses, the threshold energy of the photon if a neutron and a pion are to emerge. [*Ans.* $(m_\pi^2 + 2Mm_\pi)c^2/2M = m_\pi c^2(1 + m_\pi/2M)$]
(iv) A particle formation experiment creates reactions of the form $b+t \rightarrow b+t+n$ where b is an incident particle of mass m, t is a target of mass M at rest in the laboratory frame, and n is a new particle. Define the 'efficiency' of the experiment as the ratio of the supplied kinetic energy to the rest energy of the new particle, $m_n c^2$. Show that, at threshold, the efficiency thus defined is equal to

$$\frac{M}{m + M + m_n/2}.$$

19

19 Describe the way density and flux transform under the Lorentz transformation. Write down the continuity equation in 4-vector notation.

21 The 4-vector field F is given by $\mathsf{F} = 2\mathsf{X} + \mathsf{K}(\mathsf{X}\cdot\mathsf{X})$ where K is a constant 4-vector and $\mathsf{X} = (ct, x, y, z)$ is the 4-vector displacement in spacetime.
Evaluate the following:
(i) $\Box \cdot \mathsf{X}$
(ii) $\Box(\mathsf{X}\cdot\mathsf{X})$
(iii) $\Box^2\mathsf{X}\cdot\mathsf{X}$
(iv) $\Box \cdot \mathsf{F}$
(v) $\Box(\Box \cdot \mathsf{F})$
(vi) $\Box^2 \sin(\mathsf{K}\cdot\mathsf{X})$
[*Ans.* $4,\ 2\mathsf{X},\ 8,\ 8 + 2\mathsf{K}\cdot\mathsf{X},\ 2\mathsf{K},\ -\mathsf{K}^2\sin(\mathsf{K}\cdot\mathsf{X})$]

3 Thomas precession; Electromagnetism via 4-vectors

(**Lectures 8-12**)

Thomas precession

8 A current-carrying wire is electrically neutral in its rest frame S. The wire has cross-sectional area A and a current I flows uniformly through this cross-section. Write down the 4-vector current density in the rest frame of the wire. Obtain the 4-vector current density in the rest frame S′ of the current carriers (you may assume they all have the same charge and drift velocity). Hence show that in this frame there is a non-zero charge density in the wire. Does this imply that charge is not Lorentz invariant? Explain. Find the electric field in S′ produced by the charge density of the wire, in the region outside the wire, and show that it is the same as the field obtained by transformation of the magnetic field in frame S.

9 (i) Show that two of Maxwell's equations are guaranteed to be satisfied if the fields are expressed in terms of potentials $\mathbf{A}$, ϕ such that

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}, \qquad \mathbf{E} = -\left(\frac{\partial \mathbf{A}}{\partial t}\right) - \boldsymbol{\nabla}\phi.$$

(ii) Express the other two of Maxwell's equations in terms of $\mathbf{A}$ and ϕ.
(iii) Introduce a gauge condition to simplify the equations, and hence express Maxwell's equations in terms of 4-vectors, 4-vector operators and Lorentz scalars (a *manifestly covariant* form).

12 Give a 4-vector argument to show that the 4-vector potential of a point charge q in an arbitrary state of motion is given by

$$\mathsf{A} = \frac{q}{4\pi\epsilon_0}\frac{\mathsf{U}/c}{\mathsf{R}\cdot\mathsf{U}}$$

where U and R are suitably chosen 4-vectors which you should define in your answer.

4 Tensors and further tools in electromagnetism

(**Lectures 13-16.5**)

5 An action S defined by

$$S[q(t)] = \int_{q_1,t_1}^{q_2,t_2} \mathcal{L}(q,\dot{q},t)dt \qquad (1)$$

has an extreme (maximum or minimum) value when the path $q(t)$ satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}.$$

(i) Write down an appropriate Lagrangian function for free motion of a particle of rest mass m, and show that the canonical momentum $\partial\mathcal{L}_{\rm free}/\partial v_i = \gamma m v_i$.
(ii) Consider the Lagrangian

$$\mathcal{L}_{\rm int} = -q(\phi - \mathbf{v}\cdot\mathbf{A})$$

which describes the interaction of a particle of charge q with a field whose potentials are ϕ and $\mathbf{A}$. Obtain the canonical momenta $\partial\mathcal{L}/\partial v_i$ where $\mathcal{L} = \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int}$ is the total Lagrangian of the particle.
(iii) Write down the Euler-Lagrange equations and show that they give $(d/dt)(\gamma m\mathbf{v}) = q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B})$.
(iii) In this formalism, write down the Hamiltonian function $\mathcal{H}$ for a particle of charge q moving in a magnetic field $\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A}$. Make sure you express $\mathcal{H}$ in terms of the appropriate variables.

6 The electromagnetic field tensor $\mathbb{F}$ (sometimes called Faraday tensor) is defined such that the four-force on a charged particle is given by

$$\mathsf{F}^\mu = q\mathbb{F}^{\mu\alpha}\mathsf{U}_\alpha \qquad [\ \mathsf{F} = q\mathbb{F}\cdot\mathsf{U}$$

By comparing this to the Lorentz force equation $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$ which defines electric and magnetic fields (keeping in mind the distinction between $d\mathbf{p}/dt$ and $d\mathsf{P}/d\tau$), show that the components of the field tensor are

$$\mathbb{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}.$$

10 Show that the following two scalar quantities are Lorentz invariant:

$$\begin{aligned} D &= E^2/c^2 - B^2 \\ \alpha &= \mathbf{B} \cdot \mathbf{E}/c. \end{aligned}$$

[Hint: for the second, introduce the dual field tensor $\tilde{\mathbb{F}}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\mathbb{F}^{\alpha\beta}$].

Show that if $\alpha = 0$ but $D \neq 0$ then either there is a frame in which the field is purely electric, or there is a frame in which the field is purely magnetic. Give the condition required for each case, and find an example such frame (by specifying its velocity relative to one in which the fields are $\mathbf{E}$, $\mathbf{B}$). Suggest a type of field for which both $\alpha = 0$ and $D = 0$.

5 Energy-momentum flow; spinors and classical fields

(**Lectures 16.5-21**)

In the following group of questions, the Larmor formula for emitted power may be quoted without proof:

$$\mathcal{P}_L = -\frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c^3}\dot{\mathsf{U}} \cdot \dot{\mathsf{U}} \qquad = \frac{2}{3}\frac{q^2 a_0^2}{4\pi\epsilon_0 c^3}$$

1 (i) The electric field in a linear accelerator is 10^6 V/m. Find the power emitted by an electron traveling down the accelerator. Express your result in eV per metre assuming the electrons travel at close to the speed of light.
(ii) A magnetic field of 1 tesla is used to maintain electrons in their orbits around a synchrotron of radius 10 m. Show that the Lorentz factor of the electrons is $\gamma = 5867$. Find the radiative energy loss per revolution.
[*Ans.* 3.68×10^{-9} eV/m, 0.71 MeV]

10 A Dirac spinor

$$\Psi = \begin{pmatrix} \phi_R \\ \chi_L \end{pmatrix}$$

transforms under a change of reference frame as

$$\Psi \to \begin{pmatrix} \Lambda(v) & 0 \\ 0 & \Lambda(-v) \end{pmatrix} \Psi$$

where $\Lambda(v) = \exp\left(i\boldsymbol{\sigma} \cdot \boldsymbol{\theta}/2 - \boldsymbol{\sigma} \cdot \boldsymbol{\rho}/2\right)$
and $\Lambda(-v) = (\Lambda(v)^\dagger)^{-1} = \exp\left(i\boldsymbol{\sigma} \cdot \boldsymbol{\theta}/2 + \boldsymbol{\sigma} \cdot \boldsymbol{\rho}/2\right)$.

(i) Show that the combination

$$\left(\phi_R^\dagger, \chi_L^\dagger\right) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \phi_R \\ \chi_L \end{pmatrix} = \phi_R^\dagger \chi_L + \chi_L^\dagger \phi_R$$

is Lorentz-invariant.

(ii) How may two non-null orthogonal 4-vectors be extracted from the Dirac spinor?

12 The Klein-Gordan equation is

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = -\mu^2 c^2 \phi$$

where μ is a constant.
(i) Prove that the equation is Lorentz-covariant as long as ϕ is a Lorentz scalar field.
(ii) In the two dimensional case (i.e. one spatial dimension plus time), show that the left hand side factorizes into terms involving only first-order derivatives.
(iii) If $\mathbf{m}$ is a 3-vector, each of whose components is a constant 2×2 matrix, and $\boldsymbol{\nabla}$ is the 3-gradient operator, show that $(\mathbf{m} \cdot \boldsymbol{\nabla})^2 = \boldsymbol{\nabla}^2$ if and only if the component matrices m_i anticommute among themselves and square to 1 (i.e. $m_i^2 = I$). Identify a set of matrices with these properties.
(iv) Factorize the left hand side of the Klein-Gordan in the four-dimensional case, and hence obtain the Dirac equation. [To reduce clutter, you may find it helpful to introduce the notation $\hat{\omega} \equiv i\left(\frac{\partial}{\partial t}\right)$, $\hat{\mathbf{k}} \equiv -i\boldsymbol{\nabla}$.]
(v) Briefly discuss the plane wave solutions of the Dirac equation.

Paper BIII: Quantum, Atomic and Molecular Physics
Example questions from problem sets 1-5

David Lucas & Simon Hooker

April 2010

Problem set 1:

(a) Show that total rate of radiative transitions from an excited level to a lower level, i.e. the sum of the spontaneous and stimulated rates, can be written as:

$$R_2^{\text{rad}} = N_2 A_{21} \left[\frac{\pi^2 c^3}{\hbar \omega_{21}^3} \rho(\omega_{21}) + 1 \right]$$

(b) Hence show that this may be written in the form,

$$R_2^{\text{rad}} = N_2 A_{21} \left[\bar{n} + 1 \right]$$

where $\bar{n}$ is the mean number of photons per mode. Note, you should *not* assume that the radiation is blackbody to prove this.

(c) Use this result to explain the difference between the number of photons per mode in laser and non-laser sources.

Problem set 2:

(a) Explain what is meant by the *central field approximation* and how it leads to the concept of *configurations*. By considering the successive application of two perturbations, the residual electrostatic interaction and the spin-orbit interaction, explain the splitting of configurations into *terms* and *levels*. Hence explain what is meant by LS coupling, making clear the conditions under which it applies.

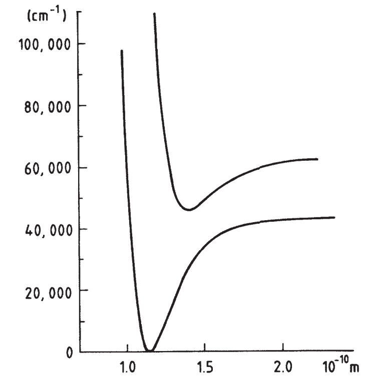
(b) Three low-lying configurations of the neutral calcium atom ($Z = 20$) are 4s^2, 4s4p and 4s5s. Write down the terms and levels which arise from these configurations, using conventional spectroscopic nomenclature. Explain the meanings of the various symbols you use. Discuss the specific mechanisms responsible for the relative ordering of these levels. Derive the *interval rule* in LS coupling, and explain whether you would expect it to be obeyed in calcium in any of the terms arising from the 4s4p and 4s5s configurations.

Problem set 3:

(a) Draw out the energy level diagram, together with the transitions, for the 671nm transition in ^6Li ($Z = 3$), $1\text{s}^2 2\text{s}\ ^2\text{S}_{1/2} \leftrightarrow 1\text{s}^2 2\text{p}\ ^2\text{P}_{3/2}$ in a weak magnetic field of magnitude B. Give the splittings of the levels in terms of $\mu_B B$ and the frequency intervals between the components in terms of $\mu_B B/h$.

(b) The levels of the $1s^2 2p\ ^2P$ term are separated by 10GHz; estimate the range of field strength which could be described as weak in this case.

Problem set 4: The figure below gives the potential energy curves for the ground state and an excited electronic state of $^{14}N^{16}O$ as a function of the inter-nuclear separation.



(a) Discuss in simple terms the form of the potential curves. Explain their main features, and indicate why the excited state curve differs from that for the ground state.

(b) In a simple first approximation the energy levels of a molecular electronic state are given by

$$E_{e,v,J} = T_e + \hbar\omega_e\left(v + \frac{1}{2}\right) + \hbar B J(J+1)$$

Explain the origin of the various terms. Estimate ω_e and B for the lower electronic state from the data in the figure.

(c) Give a simple explanation for each of the following observations concerning transitions between the excited and ground electronic states of $^{14}N^{16}O$:

i. The strongest transitions are between low-lying vibrational levels in the upper electronic state and highly excited vibrational levels of the lower electronic state.

ii. The rotational structure of some transitions shows a clustering into a 'band-head'.

Problem set 5: (a) Use a rate equation analysis to show that the gain coefficient of a homogeneously broadened laser transition is modified by the presence of narrow-band radiation of total intensity I to,

$$\alpha_I(\omega - \omega_0) = \frac{\alpha_0(\omega - \omega_0)}{1 + I/I_s(\omega_L - \omega_0)}, \tag{1}$$

where ω_L is the laser frequency. Give an expression for the saturation intensity I_s.

(b) Explain in physical terms why the saturation intensity depends on the detuning of the intense beam from the centre frequency of the transition.

(c) On the same graph plot the gain coefficient as a function of frequency ω:

i. as measured by a weak probe beam in the absence of any other radiation;

ii. as measured by a weak probe beam in the presence of a narrow-band beam of intensity $I_s(\omega_L - \omega_0)$;

iii. as measured by an intense, narrow-band beam of constant intensity $I_s(0)$.

Part B, Course IV: *Subatomic physics*

Example questions

AJB, April 21, 2010

Q1. The Yukawa potential is given by

$$V(r) = \frac{g^2}{4\pi}\frac{e^{-\mu r}}{r}.$$

a) Show that the amplitude

$$\langle k'|V|k\rangle = -\frac{g^2}{(2\pi)^3}\frac{1}{\mu^2 + \Delta k^2}.$$

b) Hence show that the first Born approximation to the differential cross-section for particle of mass m scattering from this potential is

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mg^2}{\hbar}\right)^2 \frac{1}{[2k^2(1-\cos\Theta)+\mu^2]^2}$$

where Θ is the scattering angle.

c) Find the total cross-section for the case when $\mu \neq 0$.

d) When $\mu \to 0$, $V(r) \propto 1/r$. Compare the differential cross-section above to that for classical Rutherford scattering

$$\frac{d\sigma}{d\Omega} = \frac{1}{16}\left(\frac{\alpha}{T}\right)^2 \frac{1}{\sin^4(\Theta/2)}.$$

What must be the relationship between g and α for Born approximation to reproduce the classical Rutherford formula for electron–proton scattering?

Q2. The radius R of a nucleus with mass number A is given by the formula $R = r_0 A^{\frac{1}{3}}$ with $r_0 = 1.2\,\mathrm{fm}$.

a) Use the Fermi gas model (assuming $A \approx Z$) to show that the energy ϵ_F of the Fermi level is given by

$$\epsilon_F = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}.$$

b) Estimate the total kinetic energy of the nucleons in an $^{16}\mathrm{O}$ nucleus.

c) For a nucleus with neutron number N and proton number Z the asymmetry term in the semi-empirical mass formula is

$$\frac{\gamma(N-Z)^2}{A}.$$

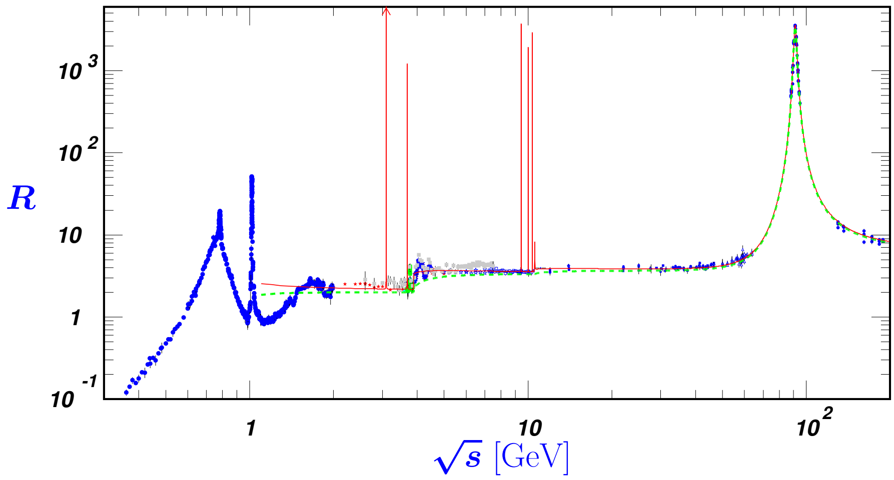
Assuming that $(N - Z) \ll A$ use the Fermi gas model to justify this form and to estimate the value of γ.

Q3. The fission of by thermal neutrons is asymmetric, the most probable mass numbers of fission fragments being 93 and 140. Use the semi-empirical mass formula to estimate the energy released in fission of $^{235}_{92}\mathrm{U}$ and hence the mass of $^{235}_{92}\mathrm{U}$ consumed each second in a 1 GW reactor. In almost all uranium ores, the proportion of $^{235}\mathrm{U}$ to $^{238}\mathrm{U}$ is 0.0072. However, in certain samples from Oklo in the Gabon the proportion is 0.0044. Assuming that a natural fission reactor operated in the Gabon 2×10^9 years ago, estimate the total energy released from 1 kg of the then naturally occurring uranium. How might the hypothesis that $^{235}\mathrm{U}$ was depleted by fission be tested?

[$t_{1/2}(^{238}\mathrm{U}) = 4.5 \times 10^9$ years, $t_{1/2}(^{235}\mathrm{U}) = 7.0 \times 10^8$ years.]

Q4. In the figure below you can find the ratio of the cross-sections for the process of electron–positron annihilation to hadrons, and the corresponding cross-section to the muon–antimuon final state as a function of $\sqrt{s}$, the center-of-mass energy.

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$



Considering the number of quarks that can be created at particular center-of-mass energy, what values of R would you expect for center-of-mass energy in the range 2 to 20 GeV? How do your predictions match the data?

What is causing the sharp peaks in R at $\sqrt{S} \approx 3\,\text{GeV}$, $10\,\text{GeV}$, and $100\,\text{GeV}$?

Q5. The Large Hadron Collider has been designed to accelerate counter-rotating beams of protons to energies of $7\,\mathrm{TeV}$, and to collide those beams at particular a small number of interaction points.

a) Use dimensional analysis to estimate the smallest length scale which this machine could be used to resolve. How does this compare to the size of e.g. atoms, nuclei and protons?

b) The LHC beam pipe is evacuated to reduce loss of beam from collisions with gas moleceules. If less than 5% of the beam protons are to be lost from collisions with gas nuclei over a ten hour run, estimate the maximum permissible number density of atoms in the pipe.

c) The machine collides counter-rotating bunches of protons, each of which has circular profile with radius $17\,\mu\mathrm{m}$ (in the direction perpendicular to travel). How many protons are required in a bunch to have an average of ten interactions per bunch crossing?

d) If such bunches collide every $25\,\mathrm{ns}$, what is the *luminosity* of the machine?

e) If the cross-section for producing a Higgs Boson is $50\,\mathrm{pb}$, how many will be made each second?

f) What is the kinetic energy of each bunch in the machine?

[Some data for proton-proton cross-sections can be found on the next page.]

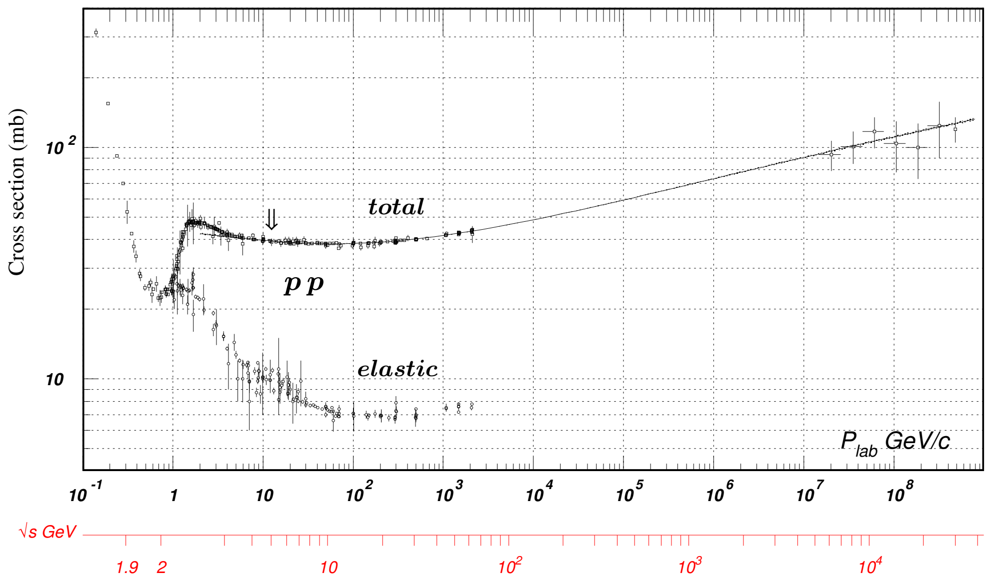


Figure 1: Proton-proton scattering cross-section data as a function of laboratory momentum. From Amsler et al. Review of Particle Physics, Particle Data Group collaboration.

Third Year General Relativity and Cosmology

Sample Problems

Prof. Pedro Ferreira

1. **Orbits in Newtonian Gravity**

   Consider a massive stationary object of mass M fixed at the origin, and a particle moving in the $x-y$ plane subject only to the Newtonian gravitational field of the massive object.

   (a) Consider the unit vectors, $\hat{\mathbf{r}}$ and $\hat{\theta}$ given in Cartesian coordinates by $\hat{\mathbf{r}} = (\cos\theta, \sin\theta)$ and $\hat{\theta} = (-\sin\theta, \cos\theta)$. Show that

   $$\frac{d}{dt}\hat{\mathbf{r}} = \dot{\theta}\hat{\theta}, \quad \frac{d}{dt}\hat{\theta} = -\dot{\theta}\hat{\mathbf{r}}$$

   (b) Writing the vector $\mathbf{r} = r\hat{\mathbf{r}}$, show that

   $$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}, \quad \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

   (c) Newton's equations for the momentum $\mathbf{p}$ of the particle of mass $m > 0$ in this situation are $\ddot{\mathbf{r}} = -(GM/r^2)\hat{\mathbf{r}}$. Show that this reduces, in components, to the two equations:

   $$\ddot{r} = r\dot{\theta}^2 + \frac{GM}{r^2} = 0, \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{1}$$

   (d) Deduce that $J = mr^2\dot{\theta}$, the angular momentum about the origin is, is constant.

   (e) By substitution, $r(t) = 1/u(\theta(t))$, show that $\dot{r} = -Jh'/m$ where $' \equiv d/d\theta$ and further that u satisfies

   $$u'' + u = \frac{GMm^2}{J^2} \tag{2}$$

   (f) Show that the general solution for the distance from the origin r, as a function of θ is

   $$r = \frac{l}{1 + e\cos(\theta - \theta_0)} \tag{3}$$

   What do these orbits look like and what is e?

2. **The Geometry around the Earth**

   Consider a particle moving on a circular orbit (of radius R) about the Earth. Assume the metric is

   $$ds^2 = -\left[1 + 2\frac{\Phi(r)}{c^2}\right]c^2dt^2 + \left[1 - 2\frac{\Phi(r)}{c^2}\right](dx^2 + dy^2 + dz^2)$$

   with $\Phi(r) = -GM_\oplus/r$. Let P be the period of the orbit measured in the time t. Consider two space-time events, A and B located at the same spatial position on the orbit but separated in t by the period P. Calculate (to first order in $1/c^2$) the proper time for an observer that

   (a) follows the orbit of the particle itself;

   (b) stays fixed in the same place throughout;

   (c) the world line of a photon that moves radially away from A and returns to B in a time P.

Compare your three results and discuss.

3. **The Einstein Tensor of the Universe**

   A spacetime has the metric

$$ds^2 = c^2 dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$

   (a) Show that the only non-zero connections coefficients are

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a\dot{a} \quad \text{and} \quad \Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30} = \dot{a}/a$$

   (b) Deduce that particles may be at rest in such a spacetime and that for such particles the coordinate t is their proper time. Show further that the non-zero components of the Ricci tensor or

$$R_{00} = 3\ddot{a}/a \quad \text{and} \quad R_{11} = R_{22} = R_{33} = -a\ddot{a} - 2\dot{a}^2$$

   (c) Hence show that the 00-component of the Einstein tensor is $G_{00} = -3\dot{a}^2/a^2$.

4. **Hubble parameter**

   Assume that the Universe is dust-dominated. Take $H_0 = 100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

   (a) Give a rough estimate of the age of the Universe.

   (b) How far can light have travelled in this time?

   (c) The microwave background has been travelling towards us uninterrupted since decoupling, when the Universe was 1/1000 of its current size. Compute the value of the Hubble parameter H at the time of decoupling.

   (d) How far could light have travelled in the time up to decoupling (assume that the Universe was dominated by radiation until then)?

   (e) Between decoupling and the present, the distance that light travelled up to the time of decoupling has been stretched by the subsequent expansion. What would be its physical size today?

   (f) Assuming that the distance to the last-scattering surface is given by part b of this question, what angle is subtended by the distance light could have travelled before decoupling?

   (g) What is the physical significance of this value?

5. **Recombination and the Surface of Last Scattering**

   (a) What is the *'surface of last scattering'*? Would the same surface be seen by any other observer on a different galaxy?

   (b) Estimate the radius of the last scattering surface, using the age of the Universe. Why might this underestimate the true value?

   (c) The present number density of electrons in the Universe is the same as that of protons, namely about 0.2 $\mathrm{m^{-3}}$. Consider a time long before decoupling when the Universe was 10^4 years old and when the scale factor was one millionth of its present value. Estimate the number density of electrons at that time and comment on whether the electrons would be relativistic or non-relativistic then.

   (d) Given that the mean free path of photons through an electron gas of number density n_e is $d \approx 1/[n_\mathrm{e}\sigma_\mathrm{e}]$, where the Thompson scattering cross-section $\sigma_\mathrm{e} = 6.7 \times 10^{-29}\mathrm{m}^2$, determine the mean free path for photons when the scale factor was one millionth its present value.

   (e) From the mean free path, calculate the typical time between interactions between the photons and electrons.

   (f) Compare the interaction time with the age of the Universe at that time. What is the significance of this comparison?

5 Sample Problems for Solid State Physics
(3rd Year Course 6)
Hilary Term 2011

1. **Velocities in the Free Electron Theory**

   (a) Assuming that the free electron theory is applicable: show that the speed v_F of an electron at the Fermi surface of a metal is $v_F = \frac{\hbar}{m}(3\pi^2 n)^{1/3}$ where n is the density of electrons.

   (b) Show that the mean drift speed v_d of an electron in an applied electric field E is $v_d = |\sigma E/(ne)|$, where σ is the electrical conductivity, and show that σ is given in terms of the mean free path λ of the electrons by $\sigma = ne^2\lambda/(mv_F)$.

   (c) Assuming that the free electron theory is applicable to copper:

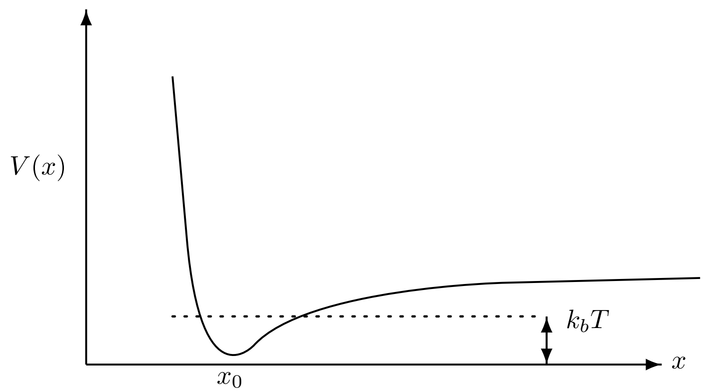
   (i) calculate the values of both v_d and v_F for copper at 300K in an electric field of 1 V m^{-1} and comment on their relative magnitudes.

   (ii) estimate λ for copper at 300K and comment upon its value compared to the mean spacing between the copper atoms.

   Copper is monovalent, meaning there is one free electron per atom. The density of atoms in copper is $n = 8.45 \times 10^{28}$ m^{-3}. The conductivity of copper is $\sigma = 5.9 \times 10^7 \Omega^{-1}\text{m}^{-1}$ at 300K.

2. **Thermal Expansion**

   As a model of thermal expansion, we study the distance between two nearest neighbor atoms in an anharmonic potential that looks roughly like this



   where x is the distance between the two neighboring atoms. This potential can be expanded around its minimum as

$$V(x) = \frac{\kappa}{2}(x - x_0)^2 - \frac{\kappa_3}{3!}(x - x_0)^3 + \ldots$$

   where the minimum is at position x_0. For small energies, we can truncate the series at the cubic term.

   (a) **Classical model:** In classical statistical mechanics, we write the expectation of x as

$$\langle x \rangle_\beta = \frac{\int dx\, x\, e^{-\beta V(x)}}{\int dx\, e^{-\beta V(x)}}$$

   Although one cannot generally do such integrals, one can expand the exponentials as

$$e^{-\beta V(x)} = e^{-\frac{\beta\kappa}{2}(x-x_0)^2}\left[1 + \frac{\beta\kappa_3}{6}(x - x_0)^3 + \ldots\right]$$

   Use this expansion to derive $\langle x \rangle_\beta$ to lowest order in κ_3, and hence show that the coefficient of thermal expansion is

$$\alpha = \frac{1}{L}\frac{dL}{dT} \approx \frac{1}{x_0}\frac{d\langle x \rangle_\beta}{dT} = \frac{1}{x_0}\frac{k_b\,\kappa_3}{2\kappa^2}$$

with k_b Boltzmann's constant. In what temperature range is the above expansion valid?

(b) **Quantum model:** In quantum mechanics we write a Hamiltonian

$$H = H_0 + V$$

where

$$H_0 = \frac{p^2}{2m} + \frac{\kappa}{2}(x - x_0)^2$$

is the Hamiltonian for the free Harmonic oscillator, and V is the perturbation

$$V = -\frac{\kappa_3}{6}(x - x_0)^3 + \ldots$$

where we will throw out quartic and higher terms. What value of m should be used here?

(i)** (Note: You can solve parts ii and iii below even if you cannot solve this part).

Use perturbation theory to show that to lowest order in κ_3,

$$\langle n|x|n\rangle = x_0 + E_n \kappa_3/(2\kappa^2) \tag{1}$$

where $|n\rangle$ is the eigenstate of the Harmonic oscillator whose energy is

$$E_n = \hbar\omega(n + \frac{1}{2}) \qquad n \geq 0$$

with $\omega = \sqrt{\kappa/m}$.

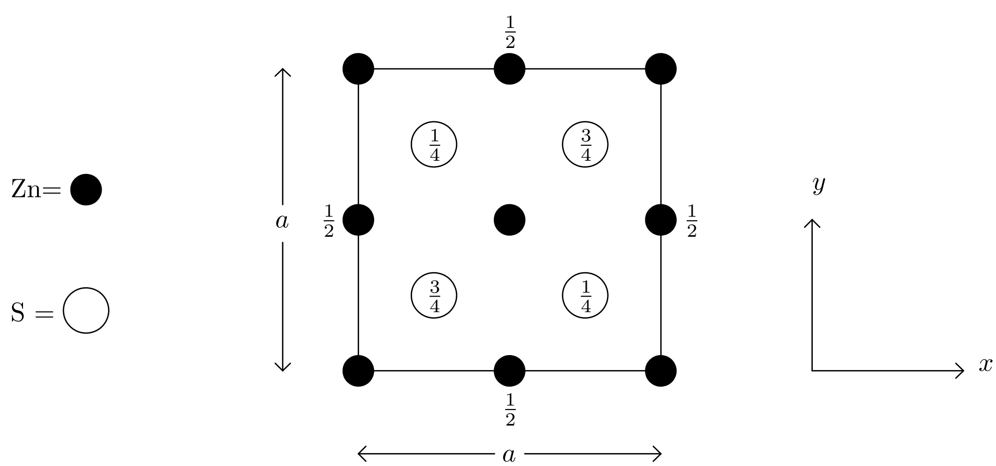
(ii) Note that even when the oscillator is in its ground state, the expectation of x deviates from x_0. Physically why is this?

(iii)* Use, Eq. 1 to calculate the quantum expectation of x at any temperature. We write

$$\langle x\rangle_\beta = \frac{\sum_n \langle n|x|n\rangle e^{-\beta E_n}}{\sum_n e^{-\beta E_n}}$$

Derive the coefficient of thermal expansion. Examine the high temperature limit and show that it matches that of part a above. In what range of temperatures is our perturbation expansion valid? In light of the current quantum calculation, when is the above classical calculation valid?

3. **Lattice Structure I**



The diagram above shows a plan view of a structure of cubic ZnS (zinc blende) looking down the z axis. The numbers attached to some atoms represent the heights of the atoms above the $z = 0$ plane expressed as a fraction of the cube edge a. Unlabeled atoms are at $z = 0$ and $z = a$.

(a) What is the Bravais lattice type

(b) Describe the basis

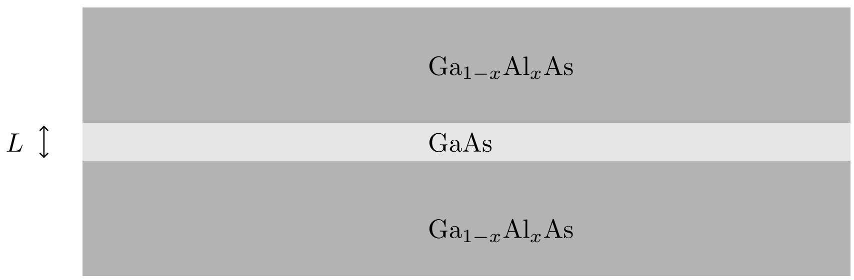
(c) Given that $a = 0.541$ nm, calculate the nearest-neighbor Zn-Zn, Zn-S, and S-S distances.

(d) Copy the drawing above, and show the [210] direction and the set of (210) planes.

(e) Calculate the spacing between adjacent (210) planes.

4. **Semiconductor Quantum Well**

A quantum well is formed from a layer of GaAs of thickness L nm, surrounded by layers of $\mathrm{Ga}_{1-x}\mathrm{Al}_x\mathrm{As}$. Sketch the shape of the potential for the electrons and holes. What approximate value of L is required if the band gap of the quantum well is to be 0.1 eV larger than that of GaAs bulk material? You may assume that the band gap of the $\mathrm{Ga}_{1-x}\mathrm{Al}_x\mathrm{As}$ is substantially larger than that of GaAs. How would it be possible to n-type dope the structure so that the electrons accumulate in a region of the structure away from the impurities? (The electron (hole) effective mass in GaAs is 0.068 $\mathrm{m_e}$ (0.45 $\mathrm{m_e}$) where $\mathrm{m_e}$ is the mass of the electron.)



5. **Mean Field Theory for the Ising Antiferromagnet**

For this exercise we use the Molecular Field approximation for the *Antiferromagnetic* Ising model on a 3 dimensional cubic lattice. The full Hamiltonian is

$$H = J \sum_{<i,j>} S_i \cdot S_j + h \sum_i S_i \tag{2}$$

Here, $S_i = \pm 1/2$ is the z component of the spin of atom on site i, $h = g\mu_B H$ is the externally applied magnetic field in the $\hat{z}$ direction, and J is the coupling between neighboring spins (with the sum indicated with $< i, j >$ means summing over i and j being neighboring sites of the cubic lattice.

For the Antiferromagnet, we have $J > 0$, so neighboring spins want to point in opposite directions. (Compared to a Ferromagnet where $J < 0$ and neighboring spins want to point in the same direction). The ordered ground state of this Hamiltonian will have alternating spins pointing up and down respectively. Let us call the sublattices of alternating sites, sublattice A and sublattice B respectively (i.e, A sites have lattice coordinates (i, j, k) with $i + j + k$ odd whereas B sites have lattice coordinates with $i + j + k$ even).

In Mean field theory (also known as Molecular Field Theory, or Weiss Mean Field Theory), the interaction between neighboring spins is replaced by an interaction with an average spin. Let the average value of the spins on lattice A be called m_A and the average of the spins on lattice B be called m_B.

(a) Derive the mean field Hamiltonians for a single site on sublattice A and the mean field Hamiltonian for a single site on sublattice B

ANSWER:

$$H_{i\,\text{in sublattice A}} = (JZm_B + h)S_i \tag{3}$$

$$H_{i\,\text{in sublattice B}} = (JZm_A + h)S_i \tag{4}$$

(Please make sure you know where this answer comes from!). What is the value of Z and what does it represent?

(b) Fixing m_A and m_B, calculate the canonical partition function for each of these sites from the Hamiltonian. Derive $\langle S_i \rangle$ in both cases, where i is on either sublattice. Derive the mean-field self consistency equations

$$m_A = -\frac{1}{2}\tanh(\beta[JZm_B + h]/2) \tag{5}$$

$$m_B = -\frac{1}{2}\tanh(\beta[JZm_A + h]/2) \tag{6}$$

with $\beta = 1/(k_b T)$.

(c) Let $h = 0$. Reduce the two self-consistency equations to a single self consistency equation. Using a graph, show that, depending on the value of βJZ there is either one solution with $m_{A,B} = 0$ (the paramagnetic phase), or three solutions with $m_A = -m_B = \pm m, 0$ (the antiferromagnetic phase). Show further that in the case where there are three possible solutions (the ferromagnetic phase) the two antiferromagnetic solutions with $m_{A,B} \neq 0$ are lower energy than the solution with $m_{A,B} = 0$.

(d) Assume $m_{A,B}$ are small near the critical point and expand the self consistency equation. Derive the critical temperature T_c below which the system is antiferromagnetic (i.e., $m_{A,B}$ become nonzero).

(e) The antiferromagnetic susceptibility can be defined as

$$\chi_{AF} = \lim_{h \to 0} \frac{\partial m_A}{\partial h} \tag{7}$$

Derive this susceptibility at arbitrary $T > T_c$ and write it in terms of T_c. Compare your result with the analogous result for a ferromagnet.