Derivation of the time dependent Hartree (Fock) equation

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Dictionary

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2. “Hartree (Fock)”: time dependent Hartree (Fock)
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Overview

1. Derivation of Hartree equations for Bosons
2. Derivation of Hartree (Fock) equations for Fermions
3. Special case: A tracer particle in the fermi sea
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Mean field for the bosons: The Hartree equation

\[ H = \sum_{j=1}^{N} -\Delta_j + \sum_{j=1}^{N} A_t(x_j) + (N - 1)^{-1} \sum_{k<j} V(x_j - x_k) \]

Interaction „felt“ by each particle of order one
\[ \Psi_0 = \prod_{j=1}^{N} \phi_0(x_j) \]
\[ id_t \Psi_t = H_t \Psi_t \]
Interaction destroys product structure.
Question:
- In which regimes: \( \Psi_t \approx \prod_{j=1}^{N} \phi_t(x_j) \) (in what sense?)
- What is \( \phi_t \)?
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- Proving $\Psi_t \approx \prod_{j=1}^{N} \phi_t(x_j)$: hard.
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Mean field for “particle 1”

\[ W(x_1) = (N - 1)^{-1} \sum_{j=2}^{N} V(x_1 - x_j) \] for fixed, \(|\phi_0|^2\)-distributed \(x_2, \ldots, x_N\).

Law of large numbers: \(|\phi_0|^2\) close to the empirical density \(\rho_0\).

\[ W(x_1) \approx V \star |\phi_0|^2(x_1) \] (“Mean field”).

Effective Dynamics: Hartree equation

\[ i d_t \phi_t = ( -\Delta + A_t + V \star |\phi_t|^2) \phi_t . \]
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Grönwall argument

Let $\alpha_t$ be a measure for the dirt in the condensate:

$$d_t\alpha_t \leq C(\alpha_t + o(1))$$

Grönwall: $\alpha_t$ stays small if $\alpha_0$ was small ($\alpha_t \leq e^{Ct}\alpha_0 + o(1)$)
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Usual approach

Control reduced one particle density matrix $\mu_1^{\Psi_t}$

$dt\mu_1^{\Psi_t}$ depends on $\mu_2^{\Psi_t}$

$dt\mu_2^{\Psi_t}$ depends on $\mu_3^{\Psi_t}$

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Technically difficult (e.g. Erdös, Schlein, Yau (2009))
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Introduction of a counting measure:

For every $j = 1, \ldots, N$ and $\phi \in L^2(\mathbb{R}^3)$ let $p^\phi_j$ be the projector given by

$$p^\phi_j = |\phi\rangle \langle \phi|_j.$$  

Let $q^\phi_j = 1 - p^\phi_j$.

According to $\Psi$ “expected relative number” of particles not in the state $\phi$

$$\alpha(\Psi, \phi) = N^{-1} \sum_{j=1}^{N} \langle \Psi, q^\phi_j \Psi \rangle = \langle \Psi, q^\phi_k \Psi \rangle = \| q^\phi_k \Psi \|^2.$$  

$$\alpha(\Psi, \phi) = 0 \iff \Psi = \prod_{j=1}^{N} \phi(x_j).$$
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$$\alpha(\Psi, \phi) = 0 \iff \Psi = \prod_{j=1}^{N} \phi(x_j).$$
Example:

\[ \Psi(x_1, \ldots, x_N) = \left( \chi(x_1, \ldots x_k) \prod_{j=k+1}^{N} \phi(x_j) \right)_{\text{sym}} \]

with \( \chi \in L^2(\mathbb{R}^{3k}) \), \( p_j \chi = 0 \) for all \( 1 \leq j \leq k \)

\[ \Rightarrow \alpha(\Psi, \phi) = k/N. \]

**Lemma:** \( \alpha(\Psi_t, \phi_t) \to 0 \) is equivalent to convergence of \( \mu_t \) to \( |\phi_t\rangle \langle \phi_t| \) in operator norm.
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**Lemma:** $\alpha(\psi_t, \phi_t) \to 0$ is equivalent to convergence of $\mu_t$ to $|\phi_t\rangle\langle\phi_t|$ in operator norm.
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Deriving the mean field equation:

Goal: show that \(|d_t \alpha_t| < C(\alpha_t + o(1)) \Rightarrow \alpha_t \ll 1\) by Grönwall

Let

\[ h_j := -\Delta_j + A + V \star |\phi_t|^2(x_j). \]

Since

\[ d_t q_j^{\phi_t} = -i[\h_j, q_j^{\phi_t}] \]

\[ d_t \alpha(\psi_t, \phi_t) = i\langle \psi_t, [H - h_1, q_1^{\phi_t}] \psi_t \rangle. \]
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\[ d_t \alpha(\Psi_t, \phi_t) = i \langle \Psi_t, [-\Delta_1 + (N - 1)^{-1} \sum_{j=2}^{N} V(x_1 - x_j) - h_1, q_{\phi_t}^1] \Psi_t \rangle \]

\[ = i \langle \Psi_t, [V(x_1 - x_2) - V \star |\phi_t|^2(x_1), q_{\phi_t}^1] \Psi_t \rangle \]

\[ = i \langle \Psi_t, (V_1(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_{\phi_t}^1 \Psi_t \rangle - c.c. \]

\[ = i \langle \Psi_t, (p_{\phi_t}^1 + q_{\phi_t}^1)(p_{\phi_t}^2 + q_{\phi_t}^2) (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_{\phi_t}^1 (p_{\phi_t}^2 + q_{\phi_t}^2) \Psi_t \rangle - c.c. \] .

All terms \( \langle \Psi_t, A_{12} \Psi_t \rangle \) where \( A_{12} \) is either selfadjoint or invariant under adjunction plus simultaneous exchange of the variables \( x_1 \) and \( x_2 \) cancel out.
\[
d_t \alpha(\Psi_t, \phi_t) = i \langle \Psi_t, [ - \Delta_1 + (N - 1)^{-1} \sum_{j=2}^N V(x_1 - x_j) - h_1, q_1^{\phi_t}] \Psi_t \rangle \\
= i \langle \Psi_t, [V(x_1 - x_2) - V \ast |\phi_t|^2(x_1), q_1^{\phi_t}] \Psi_t \rangle \\
= i \langle \Psi_t, (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^{\phi_t} \Psi_t \rangle - \text{c.c.} \\
= i \langle \Psi_t, (p_1^{\phi_t} + q_1^{\phi_t})(p_2^{\phi_t} + q_2^{\phi_t}) (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^{\phi_t} (p_2^{\phi_t} + q_2^{\phi_t}) \Psi_t \rangle - \text{c.c.}
\]

All terms \(<\Psi_t, A_{12} \Psi_t>\) where \(A_{12}\) is either selfadjoint or invariant under adjunction plus simultaneous exchange of the variables \(x_1\) and \(x_2\) cancel out.
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\[ = i \langle \Psi_t, (p_{1}^{\phi_t} + q_{1}^{\phi_t})(p_{2}^{\phi_t} + q_{2}^{\phi_t}) (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_{1}^{\phi_t} (p_{2}^{\phi_t} + q_{2}^{\phi_t}) \Psi_t \rangle - c.c. \).

All terms \( \langle \Psi_t, A_{12} \Psi_t \rangle \) where \( A_{12} \) is either selfadjoint or invariant under adjunction plus simultaneous exchange of the variables \( x_1 \) and \( x_2 \) cancel out.
\[ d_t \alpha(\psi_t, \phi_t) = i \langle \psi_t, [-\Delta_1 + (N - 1)^{-1} \sum_{j=2}^{N} V(x_1 - x_j) - h_1, q_{1t}^\phi] \psi_t \rangle \]

\[ = i \langle \psi_t, [V(x_1 - x_2) - V \ast |\phi_t|^2(x_1), q_{1t}^\phi] \psi_t \rangle \]

\[ = i \left( \langle \psi_t, (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_{1t}^\phi \psi_t \rangle - \text{c.c.} \right) \]

\[ = i \left( \langle \psi_t, (p_{1t}^\phi + q_{1t}^\phi)(p_{2t}^\phi + q_{2t}^\phi) (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_{1t}^\phi (p_{2t}^\phi + q_{2t}^\phi) \psi_t \rangle - \text{c.c.} \right). \]

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All terms \( \langle \Psi_t, A_{12} \Psi_t \rangle \) where \( A_{12} \) is either selfadjoint or invariant under adjunction plus simultaneous exchange of the variables \( x_1 \) and \( x_2 \) cancel out.
Only three types remain: $pp \ldots pq$, $pp \ldots qq$ and $pq \ldots qq$

\begin{align*}
I &= i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^{\phi_t} p_2^{\phi_t} \psi_t \rangle \\
&= i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) p_2^{\phi_t} q_1^{\phi_t} \psi_t \rangle - i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V \ast |\phi_t|^2(x_1) p_2^{\phi_t} q_1^{\phi_t} \psi_t \rangle \\
II &= i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle \\
III &= i \langle \psi_t, p_1^{\phi_t} q_2^{\phi_t} (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle.
\end{align*}
Only three types remain: \( pp \ldots pq \), \( pp \ldots qq \) and \( pq \ldots qq \)

\[
I = i\langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle \\
= i\langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle - i\langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} V \star |\phi_t|^2(x_1) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle \\
II = i\langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle \\
III = i\langle \Psi_t, p_1^{\phi_t} q_2^{\phi_t} (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle .
\]
Only three types remain: \( pp \ldots pq, \ pp \ldots qq \) and \( pq \ldots qq \)

\[
I = i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} \left( V(x_1 - x_2) - V \ast |\phi_t|^2(x_1) \right) q_1^{\phi_t} p_2^{\phi_t} \psi_t \rangle \\
= i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) p_2^{\phi_t} q_1^{\phi_t} \psi_t \rangle - i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V \ast |\phi_t|^2(x_1) p_2^{\phi_t} q_1^{\phi_t} \psi_t \rangle \\
II = i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle \\
III = i \langle \psi_t, p_1^{\phi_t} q_2^{\phi_t} \left( V(x_1 - x_2) - V \ast |\phi_t|^2(x_1) \right) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle .
\]
Only three types remain: $pp \ldots pq$, $pp \ldots qq$ and $pq \ldots qq$

\[
I = i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle
\]
\[
= i \langle \psi_t, p_1^{\phi_t} q_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} \psi_t \rangle - i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V \ast |\phi_t|^2(x_1) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle
\]

\[
II = i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} (V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t}) \psi_t \rangle
\]

\[
III = i \langle \psi_t, p_1^{\phi_t} q_2^{\phi_t} (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle.
\]
Only three types remain: $pp \ldots pq$, $pp \ldots qq$ and $pq \ldots qq$

egin{align*}
I &= i \langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^{\phi_t} p_2^{\phi_t} \Psi_t \rangle \\
   &= i \langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) p_2^{\phi_t} q_1^{\phi_t} \Psi_t \rangle \quad - \quad i \langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} V \star |\phi_t|^2(x_1) p_2^{\phi_t} q_1^{\phi_t} \Psi_t \rangle \\
II &= i \langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t} \Psi_t \rangle \\
III &= i \langle \Psi_t, p_1^{\phi_t} q_2^{\phi_t} (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \Psi_t \rangle.
\end{align*}
Only three types remain: \( pp \ldots pq, pp \ldots qq \) and \( pq \ldots qq \)

\[
I = i \langle \psi_t, p_1^\phi t p_2^\phi t (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^\phi t p_2^\phi t \psi_t \rangle \\
= i \langle \psi_t, p_1^\phi t p_2^\phi t V(x_1 - x_2) p_2^\phi t q_1^\phi t \psi_t \rangle - i \langle \psi_t, p_1^\phi t p_2^\phi t V \star |\phi_t|^2(x_1) p_2^\phi t q_1^\phi t \psi_t \rangle
\]

\[
II = i \langle \psi_t, p_1^\phi t p_2^\phi t V(x_1 - x_2) q_1^\phi t q_2^\phi t \psi_t \rangle
\]

\[
III = i \langle \psi_t, p_1^\phi t q_2^\phi t (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^\phi t q_2^\phi t \psi_t \rangle.
\]

\[
I = p_2^\phi t V(x_1 - x_2) p_2^\phi t
\]
Only three types remain: $pp \ldots pq$, $pp \ldots qq$ and $pq \ldots qq$

\[ I = i \langle \psi_t, p_1^\phi_t p_2^\phi_t (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^\phi_t p_2^\phi_t \psi_t \rangle \]

\[ = i \langle \psi_t, p_1^\phi_t p_2^\phi_t V(x_1 - x_2) p_2^\phi_t q_1^\phi_t \psi_t \rangle - i \langle \psi_t, p_1^\phi_t p_2^\phi_t V \ast |\phi_t|^2(x_1) p_2^\phi_t q_1^\phi_t \psi_t \rangle \]

\[ II = i \langle \psi_t, p_1^\phi_t p_2^\phi_t V(x_1 - x_2) q_1^\phi_t q_2^\phi_t \psi_t \rangle \]

\[ III = i \langle \psi_t, p_1^\phi_t q_2^\phi_t (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^\phi_t q_2^\phi_t \psi_t \rangle . \]

\[ I : p_2^\phi_t V(x_1 - x_2) p_2^\phi_t = |\phi_t(x_2)\rangle \langle \phi_t(x_2)| V(x_1 - x_2) |\phi_t(x_2)\rangle \langle \phi_t(x_2)| \]
Only three types remain: \( pp \ldots pq \), \( pp \ldots qq \) and \( pq \ldots qq \)

\[
I = i \langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^{\phi_t} p_2^{\phi_t} \psi_t \rangle
\]

\[
= i \langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) p_2^{\phi_t} q_1^{\phi_t} \psi_t \rangle - i \langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} V \star |\phi_t|^2(x_1) p_2^{\phi_t} q_1^{\phi_t} \psi_t \rangle
\]

\[
II = i \langle \Psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle
\]

\[
III = i \langle \Psi_t, p_1^{\phi_t} q_2^{\phi_t} (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle.
\]

\[
I : p_2^{\phi_t} V(x_1 - x_2) p_2^{\phi_t} = |\phi_t(x_2)\rangle \langle \phi_t(x_2)| V(x_1 - x_2) |\phi_t(x_2)\rangle \langle \phi_t(x_2)|
\]

\[
= |\phi_t(x_2)\rangle V \star |\phi_t|^2(x_1) |\phi_t(x_2)\rangle = p_2^{\phi_t} V \star |\phi_t|^2
\]

\[
= p_2^{\phi_t} V \star |\phi_t|^2 p_2^{\phi_t}
\]

\[
\Rightarrow I = 0.
\]
Only three types remain: \(pp \ldots pq\), \(pp \ldots qq\) and \(pq \ldots qq\)

\[
I = i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^{\phi_t} p_2^{\phi_t} \psi_t \rangle \\
= i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) p_2^{\phi_t} q_1^{\phi_t} \psi_t \rangle - i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V \star |\phi_t|^2(x_1) p_2^{\phi_t} q_1^{\phi_t} \psi_t \rangle \\
II = i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle \\
III = i \langle \psi_t, p_1^{\phi_t} q_2^{\phi_t} (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle.
\]

\[
I : p_2^{\phi_t} V(x_1 - x_2) p_2^{\phi_t} = |\phi_t(x_2) \rangle \langle \phi_t(x_2)| V(x_1 - x_2) |\phi_t(x_2) \rangle \langle \phi_t(x_2)| \\
= |\phi_t(x_2) \rangle V \star |\phi_t|^2(x_1) |\phi_t(x_2) \rangle = p_2^{\phi_t} V \star |\phi_t|^2 \\
= p_2^{\phi_t} V \star |\phi_t|^2 p_2^{\phi_t}
\]

\[\Rightarrow I = 0.\]
Only three types remain: $pp \ldots pq$, $pp \ldots qq$ and $pq \ldots qq$

\[
I = i \langle \psi_t, p_1^\phi t p_2^\phi t (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^\phi t p_2^\phi t \psi_t \rangle
\]

\[
= i \langle \psi_t, p_1^\phi t p_2^\phi t V(x_1 - x_2) p_2^\phi t q_1^\phi t \psi_t \rangle - i \langle \psi_t, p_1^\phi t p_2^\phi t V \ast |\phi_t|^2(x_1) p_2^\phi t q_1^\phi t \psi_t \rangle
\]

\[
II = i \langle \psi_t, p_1^\phi t p_2^\phi t V(x_1 - x_2) q_1^\phi t q_2^\phi t \psi_t \rangle
\]

\[
III = i \langle \psi_t, p_1^\phi t q_2^\phi t (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^\phi t q_2^\phi t \psi_t \rangle.
\]

\[
I : p_2^\phi t V(x_1 - x_2) p_2^\phi t = |\phi_t(x_2)\rangle \langle \phi_t(x_2)| V(x_1 - x_2) |\phi_t(x_2)\rangle \langle \phi_t(x_2)|
\]

\[
= |\phi_t(x_2)\rangle V \ast |\phi_t|^2(x_1) |\phi_t(x_2)\rangle = p_2^\phi t V \ast |\phi_t|^2
\]

\[
= p_2^\phi t V \ast |\phi_t|^2 p_2^\phi t
\]

\[
\Rightarrow I = 0.
\]
Only three types remain: \( pp\ldots pq, \ pp\ldots qq \) and \( pq\ldots qq \)

\[
I = i \langle \psi_t, \ p_1^\phi_t \ p_2^\phi_t \ (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) \ q_1^\phi_t \ q_2^\phi_t \Psi_t \rangle \\
= i \langle \psi_t, \ p_1^\phi_t \ p_2^\phi_t \ V(x_1 - x_2) \ q_1^\phi_t \ q_2^\phi_t \Psi_t \rangle - i \langle \psi_t, \ p_1^\phi_t \ p_2^\phi_t \ V \star |\phi_t|^2(x_1) \ q_1^\phi_t \ q_2^\phi_t \Psi_t \rangle \\
II = i \langle \psi_t, \ p_1^\phi_t \ p_2^\phi_t \ V(x_1 - x_2) \ q_1^\phi_t \ q_2^\phi_t \Psi_t \rangle \\
III = i \langle \psi_t, \ p_1^\phi_t \ q_2^\phi_t \ (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) \ q_1^\phi_t \ q_2^\phi_t \Psi_t \rangle .
\]

\[
I : p_2^\phi_t \ V(x_1 - x_2) p_2^\phi_t = |\phi_t(x_2)\rangle \langle \phi_t(x_2)| V(x_1 - x_2) |\phi_t(x_2)\rangle \langle \phi_t(x_2)| \\
= |\phi_t(x_2)\rangle V \star |\phi_t|^2(x_1) |\phi_t(x_2)\rangle = p_2^\phi_t V \star |\phi_t|^2 \\
= p_2^\phi_t V \star |\phi_t|^2 p_2^\phi_t
\]

\[
\Rightarrow I = 0.
\]
Only three types remain: \( pp \ldots pq, \ pp \ldots qq \) and \( pq \ldots qq \)

\[
I = i \langle \Psi_t, p_1^\phi p_2^\phi (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^\phi q_2^\phi \Psi_t \rangle
\]

\[
= i \langle \Psi_t, p_1^\phi p_2^\phi V(x_1 - x_2) q_1^\phi q_2^\phi \Psi_t \rangle - i \langle \Psi_t, p_1^\phi p_2^\phi V \star |\phi_t|^2(x_1) q_2^\phi \Psi_t \rangle
\]

\[
II = i \langle \Psi_t, p_1^\phi p_2^\phi V(x_1 - x_2) q_1^\phi q_2^\phi \Psi_t \rangle
\]

\[
III = i \langle \Psi_t, p_1^\phi q_2^\phi (V(x_1 - x_2) - V \star |\phi_t|^2(x_1)) q_1^\phi q_2^\phi \Psi_t \rangle
\].

\[
I : p_2^\phi V(x_1 - x_2) p_2^\phi = \langle \phi_t(x_2) | V(x_1 - x_2) | \phi_t(x_2) \rangle \langle \phi_t(x_2) | V(x_1 - x_2) | \phi_t(x_2) \rangle
\]

\[
= \langle \phi_t(x_2) | V \star |\phi_t|^2(x_1) | \phi_t(x_2) \rangle = p_2^\phi V \star |\phi_t|^2
\]

\[
= p_2^\phi V \star |\phi_t|^2 p_2^\phi
\]

\[\Rightarrow I = 0.\]
Only three types remain: $pp\ldots pq$, $pp\ldots qq$ and $pq\ldots qq$

$I = 0$

$II = i \langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle$

$III = i \langle \psi_t, q_2^{\phi_t} p_1^{\phi_t} (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle$.

\[ |III| \leq \|q_2^{\phi_t} \psi_t\|^2 2\|V\|_\infty = C\alpha(\psi_t, \phi_t) \]

\[ d_t \alpha(\psi_t, \phi_t) \leq C(\alpha(\psi_t, \phi_t) + N^{-1}) \]
Only three types remain: \( pp \ldots pq, \ pp \ldots qq \) and \( pq \ldots qq \)

\[ I = 0 \]

\[ II = i\langle \psi_t, p_1^{\phi_t} p_2^{\phi_t} V(x_1 - x_2) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle \]

\[ III = i\langle \psi_t, q_2^{\phi_t} p_1^{\phi_t} (V(x_1 - x_2) - V \ast |\phi_t|^2(x_1)) q_1^{\phi_t} q_2^{\phi_t} \psi_t \rangle. \]

\[ |III| \leq \|q_2^{\phi_t} \psi_t\|^2 2\|V\|_\infty = C \alpha(\psi_t, \phi_t) \]

\[ d_t \alpha(\psi_t, \phi_t) \leq C(\alpha(\psi_t, \phi_t) + N^{-1}) \]
Fermions

- Microscopic system: \( \Psi_0 = \Lambda_{j=1}^N \phi_j^0 \)

\[
H = \sum_{j=1}^N -\Delta_j + \sum_{j=1}^N A_t(x_j) + N^{-2/3} \sum_{k<j} V(x_j - x_k)
\]

\( N^{-2/3} \)-coupling: volume \( N \), Coulomb interaction

\[
\int_0^{N^{4/3}} x^{-1} d^3 x \sim N^{2/3}
\]

- Macroscopic equation: Hartree (Fock)

\[
d_t \phi_t^j = \left( -\Delta + A + N^{-1} \sum_{k=1}^N |\phi_t^k|^2 \ast V \right) \phi_t^j
\]

\[
- \left( N^{-1} \sum_{k=1}^N \left( \phi_t^k \ast \phi_t^j \right) \ast V \right) \phi_t^k
\]

- \( p = \sum_{j=1}^N |\phi_t^j \rangle \langle \phi_t^j | \)

- Difficulties: \( I \neq 0 \)

\( N^{-2/3} \) instead of \( N^{-1} \)
Fermions

- Microscopic system: \( \Psi_0 = \Lambda_{j=1}^N \phi_j^0 \)

\[
H = \sum_{j=1}^N -\Delta_j + \sum_{j=1}^N A_t(x_j) + N^{-2/3} \sum_{k<j} V(x_j - x_k)
\]

\( N^{-2/3} \)-coupling: volume \( N \), Coulomb interaction

\[
\int_0^{N^{1/3}} x^{-1} d^3 x \sim N^{2/3}
\]

- Macroscopic equation: Hartree (Fock)

\[
d_t \phi_t^j = \left( -\Delta + A + N^{-1} \sum_{k=1}^N |\phi_t^k|^2 \ast V \right) \phi_t^j
\]

- \( p = \sum_{j=1}^N |\phi_t^j\rangle\langle \phi_t^j| \)

- Difficulties: \( I \neq 0 \)

\( N^{-2/3} \) instead of \( N^{-1} \)
Fermions

- Microscopic system: $\Psi_0 = \Lambda^{j=1}_j \phi^j_0$
  
  \[ H = \sum_{j=1}^N -\Delta_j + \sum_{j=1}^N A_t(x_j) + N^{-2/3} \sum_{k<j} V(x_j - x_k) \]
  
  $N^{-2/3}$-coupling: volume $N$, Coulomb interaction
  
  \[ \int_0^{N^{1/3}} x^{-1} d^3 x \sim N^{2/3} \]

- Macroscopic equation: Hartree (Fock)
  
  \[ d_t \phi^j_t = \left( -\Delta + A + N^{-1} \sum_{k=1}^N |\phi^k_t|^2 \ast V \right) \phi^j_t \]
  
  \[ - \left( N^{-1} \sum_{k=1}^N (\phi^k_t \ast \phi^j_t) \ast V \right) \phi^k_t \]

- $p = \sum_{j=1}^N |\phi^j_t\rangle \langle \phi^j_t|$}

- Difficulties: $I \neq 0$
  
  $N^{-2/3}$ instead of $N^{-1}$

Derivation of the time dependent Hartree (Fock) equation
Fermions

- Microscopic system: \( \Psi_0 = \Lambda_{j=1}^N \phi_j^0 \)
  
  \[ H = \sum_{j=1}^N -\Delta_j + \sum_{j=1}^N A_t(x_j) + N^{-2/3} \sum_{k<j} V(x_j - x_k) \]

  \( N^{-2/3} \)-coupling: volume \( N \), Coulomb interaction

\[ \int_0^{N^{1/3}} x^{-1} d^3 x \sim N^{2/3} \]

- Macroscopic equation: Hartree (Fock)
  
  \[ d_t \phi_t^j = \left( -\Delta + A + N^{-1} \sum_{k=1}^N |\phi_t^k|^2 \star V \right) \phi_t^j \]

  \[ \quad - \left( N^{-1} \sum_{k=1}^N \left( \phi_t^k \star \phi_t^* \right) \star V \right) \phi_t^k \]

- \( p = \sum_{j=1}^N |\phi_t^j\rangle \langle \phi_t^j| \)

- Difficulties: \( I \neq 0 \)
  
  \( N^{-2/3} \) instead of \( N^{-1} \)
Fermions

- Microscopic system: \( \Psi_0 = \Lambda_{j=1}^N \phi_j^0 \)

\[
H = \sum_{j=1}^N -\Delta_j + \sum_{j=1}^N A_t(x_j) + N^{-2/3} \sum_{k<j} V(x_j - x_k)
\]

\(N^{-2/3}\)-coupling: volume \( N \), Coulomb interaction

\[
\int_0^{N^{1/3}} x^{-1} d^3 x \sim N^{2/3}
\]

- Macroscopic equation: Hartree (Fock)

\[
d_t \phi^j_t = \left( -\Delta + A + N^{-1} \sum_{k=1}^N |\phi^k_t|^2 \ast V \right) \phi^j_t
\]

\[- \left( N^{-1} \sum_{k=1}^N \left( \phi^k_t \ast \phi^j_t \right) \ast V \right) \phi^k_t \]

- \( p = \sum_{j=1}^N |\phi^j_t\rangle \langle \phi^j_t| \)

- Difficulties: \( I \neq 0 \)

\( N^{-2/3} \) instead of \( N^{-1} \)
Fermions

- Microscopic system: \( \Psi_0 = \Lambda_{j=1}^N \phi_j^0 \)
  \[
    H = \sum_{j=1}^N -\Delta_j + \sum_{j=1}^N A_t(x_j) + N^{-2/3} \sum_{k<j} V(x_j - x_k)
  \]
  \( N^{-2/3} \) - coupling: volume \( N \), Coulomb interaction

\[
  \int_0^{N^{1/3}} x^{-1} d^3 x \sim N^{2/3}
\]

- Macroscopic equation: Hartree (Fock)
  \[
  d_t \phi_t^j = \left( -\Delta + A + N^{-1} \sum_{k=1}^N |\phi_t^k|^2 \ast V \right) \phi_t^j
  - \left( N^{-1} \sum_{k=1}^N \left( \phi_t^k \ast \phi_t^j \ast V \right) \phi_t^k \right)
  - \left( N^{-1} \sum_{k=1}^N \left( \phi_t^j \ast \phi_t^k \ast V \right) \phi_t^j \right)
  \]
  \[
  p = \sum_{j=1}^N |\phi_t^j \rangle \langle \phi_t^j |
  \]
  \( N^{-2/3} \) instead of \( N^{-1} \)

Difficulties: \( I \neq 0 \)
Fermions

- Microscopic system: \( \Psi_0 = \Lambda_{j=1}^N \phi_j^0 \)
  \[ H = \sum_{j=1}^N -\Delta_j + \sum_{j=1}^N A_t(x_j) + N^{-2/3} \sum_{k<j} V(x_j - x_k) \]
  \( N^{-2/3} \)-coupling: volume \( N \), Coulomb interaction
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- Macroscopic equation: Hartree (Fock)
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  \[ - \left( N^{-1} \sum_{k=1}^N \left( \phi_k^t, \phi_j^t \right) \ast V \right) \phi_k^t \]

- \( p = \sum_{j=1}^N \phi_j^t \langle \phi_j^t \rangle \)

- Difficulties: \( l \neq 0 \)
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Fermions

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- Semiclassical situation: N. Benedikter, M. Porta, B. Schlein (2014)
- Problem: potential of leading order, force small!
- Goal: Consider coupling $N^{-1/3}$. 
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Peter Pickl
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Derivation of the time dependent Hartree (Fock) equation
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Special situation

- Tracer in filled Fermi sea: $\psi_0 = \chi(y) \wedge_{j=1}^{N} \phi_j(x_j)$
- Interaction with tracer and gas particles: $H_I = \sum_{j=1}^{N} V(y, x_j)$
- Empirics: free evolution of tracer
- Strong contrast to bosonic case $\psi_0 = \chi(y) \left( \prod_{j=1}^{N} \phi_j(x_j) \right)_{\text{sym}}$
- Brownian motion.
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Special situation

1d: easy: Momentum and energy conservation.
Special situation

Higher dimensions

![Graph showing energy versus total momentum](image-url)
Estimate of fluctuations

Variance of force at some position $y$
(fermions: purble, bosons: blue)
Estimate of fluctuations

Variance of force at some position $y$
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\begin{figure}
\centering
\includegraphics[width=\textwidth]{chart.png}
\end{figure}
Estimate of fluctuations

Variance of force at some position $y$
(fermions: purble, bosons: blue)

- Fluctuation of force is much smaller for fermions, still large
- Correlation due to antisymmetry reduces fluctuations.
- Fluctuations caused by particles with high momentum
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Thank you!