Entanglement Spectroscopy and its application to the fractional quantum Hall phases

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Topological phases

What is topological order?

- phases that can’t be described by a broken symmetry. No local order parameter.
- At least one physical (i.e. measurable) quantity related to a topological invariant (like the surface genus).
- A system with a gapped bulk and gapped or gapless surface (or edge) modes.
- Simplest example: the integer quantum Hall effect (quantized Hall conductance).

Since 2005, the revolution of topological insulators
Making things harder: strong interactions

- Most celebrate example: the fractional quantum Hall effect.
- Alliance of a non-trivial band structure (Landau levels) and strong interactions.
- An exotic place: emergent fractional charges with fractional statistics or non-abelian.

- No classification of fractional phases (as opposed to the non-interacting case).
- Non-perturbative problem → variational methods and numerical simulations.
- No local order parameter → which phase is emerging?
Outline

- Entanglement Spectrum
- Fractional Quantum Hall Effect
- FQHE and Entanglement Spectrum
- ES and Fractional Chern Insulators
Entanglement Spectrum
Entanglement spectrum (Li and Haldane 2008)

- Start from a quantum state $|\Psi\rangle$.
- Create a bipartition of the system into $A$ and $B$.
- Reduced density matrix
  \[ \rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi| \]
- Entanglement Hamiltonian:
  \[ \rho_A = e^{-H_{\text{ent}}} \]

  - The eigenvalues of $H_{\text{ent}}$ are the entanglement energies $\{\xi_i\}$.
  - Lower entanglement energies $\simeq$ higher weights in $\rho_A$.
  - If $O = O_A + O_B$ and , the $\xi_i$ can be labeled by the $O_A$ quantum numbers.
  - Entanglement entropy $S_A = -\text{Tr}_A [\rho_A \ln \rho_A]$, area law for gapped systems (i.e. $S_A \propto L^{d-1}$).
Entanglement spectrum

Example: system made of two spins 1/2

Entanglement spectrum: $\xi$ as a function of $S_{z,A}$ (z projection of the spin A)

The counting (i.e., the number of non-zero eigenvalues) also provides information about the entanglement.
The AKLT spin chain

A prototype of a gapped spin-1 chain.

\[ H_{\text{AKLT}} = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} \sum_j \left( \vec{S}_j \cdot \vec{S}_{j+1} \right)^2 \]

The ground state of the AKLT Hamiltonian is the valence bond state.

For an open chain, the two extreme unpaired spin-\( \frac{1}{2} \) correspond to the edge excitations (4-fold degenerate ground state)
The AKLT spin chain, the Li-Haldane conjecture

- Reduced density matrix is $81 \times 81$ but only two non-zero eigenvalues for the AKLT model.
- The trace has introduced an artificial edge $\rightarrow$ a spin-$\frac{1}{2}$ edge excitation. The ES mimics the edge spectrum of the model.
- Away from the model state: An entanglement gap $\Delta_\xi$ between a low (entanglement) energy structure related the model state and a high energy structure. $\Delta_\xi$ should stay finite at the thermodynamical limit if the two phases are in the same universality class.

ES for an open chain with 8 sites and $l_A = 4$. $S_{z,A}$: $z$-projection of $A$ total spin.
Fractional Quantum Hall Effect
Fractional Quantum Hall effect

Landau levels (spinless case)

- Cyclotron frequency: $\omega_c = \frac{eB}{m}$
- Filling factor: $\nu = \frac{\hbar n}{eB} = \frac{N}{N\Phi}$
- Partial filling + interaction $\rightarrow$ FQHE
- Lowest Landau level ($\nu < 1$):
  $z^m \exp \left(-\frac{|z|^2}{4\ell_B^2}\right)$
- $N$-body wave function:
  $\Psi = P(z_1, \ldots, z_N) \exp(-\sum |z_i|^2/(4\ell_B^2))$
- What are the low energy properties? Gapped bulk, Massless edge
- Strongly correlated systems, emergence of exotic phases: fractional charges, non-abelian braiding.

\[ \begin{align*}
N=2 & \quad \hbar \omega_c \\
N=1 & \quad \hbar \omega_c \\
N=0 & \quad \hbar \omega_c \\
\end{align*} \]
The Fractional QHE

FQHE is a hard $N$-body problem:

- a single Landau level (the lowest one for $\nu < 1$, no spin)
- the effective Hamiltonian is just the (projected) interaction!

$$\mathcal{H} = \mathcal{P}_{LLL} \sum_{i<j} V(\vec{r}_i - \vec{r}_j) \mathcal{P}_{LLL}$$

(insert in $V$ your favorite interaction plus screening effect, finite width, Landau level,...)

Two major methods:

- variational method: find a wave functions describing low energy physics (symmetries, CFT, model...)
- numerical calculation: exact diagonalizations on different geometries (sphere, plane, torus, ...), DMRG

$\text{Nbr orb.} \approx N_\Phi$
The Laughlin wave function

A (very) good approximation of the ground state at $\nu = \frac{1}{3}$

$$\Psi_L(z_1, ... z_N) = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i \frac{|z_i|^2}{4j^2}}$$

- The Laughlin state is the unique (on genus zero surface) densest state that screens the short range (p-wave) repulsive interaction.
- **Topological state**: the degeneracy of the densest state depends on the surface genus (sphere, torus, ...
The Laughlin wave function: quasihole excitations

Add one flux quantum at $z_0 = \text{one quasi-hole}$

$$\Psi_{qh}(z_1, \ldots z_N) = \prod_i (z_0 - z_i) \, \Psi_L(z_1, \ldots z_N)$$

- Locally, create one quasi-hole with fractional charge $\pm \frac{e}{3}$
- Quasi-holes obey fractional statistics (fractional charge + flux)
- Adding quasiholes/flux quanta increases the size of the droplet
- For given number of particles and flux quanta, there is a specific number of qh states that one can write
- These numbers/degeneracies can be classified with respect some quantum number (angular momentum $L_z$) and are a fingerprint of the phase (related to the statistics of the excitations).
In the LLL, the one-body wf are:
\[ \sum_{\mathbf{k} \in \mathbb{Z}} e^{\frac{2\pi}{L_y} (k_y + k_N \phi)(x + iy)} e^{-\frac{x^2}{2}} e^{-\frac{1}{2} \left( \frac{2\pi}{L_y} \right)^2 (k_y + k_N \phi)^2} \]

- The Laughlin \( \nu = 1/m \) is \( m \)-fold degenerate on the torus.
- Number of orbitals is \( N_{\phi} \).
- Each orbital is labeled by its quantum number \( k_y \).
- Invariant under the magnetic translations.
- \( K_T^y = (\sum_i k_y, i) \mod N_{\phi} \).
- There is another quantum number (purely many-body) related the center of mass degeneracy.

Model interaction for the Laughlin state, 
\( N = 6, N_{\phi} = 18 \)
The Haldane’s exclusion principle

- The number of quasihole states per momentum sector can be predicted by a generalization of the Pauli’s principle.
- For the Laughlin $\nu = 1/m$, no more than 1 particle in $m$ consecutive orbitals (including periodic boundary conditions on the torus).
- Example Laughlin $\nu = 1/3$ state with 9 flux quanta

  $k_y = 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8$
  
  1 0 0 1 0 0 1 0 0  \[ \checkmark \]
  
  1 0 0 0 1 0 1 0 0  \[ \times \]

  $k_y = 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8$
  
  1 0 0 0 1 0 0 1 0  \[ \times \]
  
  0 1 0 0 1 0 0 1 0  \[ \checkmark \]

- Can be generalized to the Moore-Read or Read-Rezayi states (non-abelian excitations): no more than $k$ particles in $k + 2$ consecutive orbitals.
**Example**: Finding back the 3-fold degeneracy of the Laughlin $\nu = 1/3$ with $N_\phi = 3 \times N = 9$

\[
k_y = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}
\]

$K_T^y = 0 + 3 + 6 \mod 9 = 0$

\[
k_y = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}
\]

$K_T^y = 1 + 4 + 7 \mod 9 = 3$

\[
k_y = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}
\]

$K_T^y = 2 + 5 + 8 \mod 9 = 6$
The Laughlin wave function: edge excitations

- One dimensional chiral mode with a linear dispersion relation $E \sim \frac{2\pi v}{L} n$
- The degeneracy of each many-body energy level $E_t$ is given by the sequence $1, 1, 2, 3, 5, 7, \ldots$
The Laughlin wave function: edge excitations

- One dimensional chiral mode with a linear dispersion relation \( E \approx \frac{2\pi \nu}{L} n \)
- The degeneracy of each many-body energy level \( E_t \) is given by the sequence 1, 1, 2, 3, 5, 7, ....

**Interacting case (Laughlin \( \nu = \frac{1}{3} \))**

- (a) \( E_t = 0 \)
- (b) \( E_t = 1 \)
- (c) \( E_t = 1 \)
- (d) \( E_t = 2 \)
- (e) \( E_t = 2 \)
FQHE and Entanglement Spectrum
Orbital entanglement spectrum

- FQHE on a cylinder (Landau gauge): orbitals are labeled by $k_y$, rings at position $\frac{2\pi k_y}{L} l_B^2$
- Divide your orbitals into two groups $A$ and $B$, keeping $N_{\text{orb},A}$ orbitals: orbital cut $\sim$ real space cut (fuzzy cut)

![Diagram showingorbital entanglement spectrum](image)

- Fingerprint of the edge mode (edge mode counting) can be read from the ES. ES mimics the chiral edge mode spectrum.
- For FQH model states, nbr. levels is exp. lower than expected.
Different eigenvalues of $\rho_A$ (shape of the ES) but the same number of non-zero eigenvalues (counting)

The counting IS the important feature. For model states in the FQHE, exponentially lower than expected
Away from model states

Groundstate of the Coulomb interaction at \( \nu = 1/3 \) for \( N = 12 \) on a sphere/thin annulus

\( \nu = 1/3 \) Laughlin state

- A low ent. energy structure identical to the Laughlin state.
- An entanglement gap that does not spread over the full spectrum but protects the region mimicking the edge mode.
- An additional structure in the high energy part related to the neutral excitations (quasihole-quasielectron pairs).
How to cut the system?

The system can be cut in different ways:
- real space
- orbital (or momentum) space
- particle space

Each way may provide different information about the system (ex: trivial in momentum space but not in real space)

- **Real space partitioning**: extracting the edge physics
- **Particle partitioning**: extracting the bulk physics
Particle entanglement spectrum

Ground state $\Psi$ for $N$ particles, remove $N - N_A$, keep $N_A$

$$\rho_A(x_1, \ldots, x_{N_A}; x'_1, \ldots, x'_{N_A}) = \int \cdots \int dx_{N_A+1} \cdots dx_N$$

$$\psi^*(x_1, \ldots, x_{N_A}, x_{N_A+1}, \ldots, x_N) \times \psi(x'_1, \ldots, x'_{N_A}, x_{N_A+1}, \ldots, x_N)$$

$\nu = 1/3$ Laughlin $N = 8$, $N_A = 4$

Coulomb GS at $\nu = 1/3$ on a torus

Counting is the number of quasihole states for $N_A$ particles on the same geometry $\rightarrow$ the fingerprint of the phase.
ES and Fractional Chern Insulators
A Chern insulator is a zero magnetic field version of the QHE (Haldane, 88).

Topological properties emerge from the band structure.

At least one band is a non-zero Chern number $C$, Hall conductance $\sigma_{xy} = \frac{e^2}{h} |C|$

What about the strong interacting regime? → Fractional Chern insulators.

Filling the lowest band $\nu = 1/3$ plus strong interaction, do we get a Laughlin-like state or a charge density wave?

Energy spectrum with similar features (3-fold degenerate groundstate) but a different entanglement spectrum.
Conclusion

- For many quantum phases, the ground state contains a surprisingly large amount of information about the excitations.
- The entanglement spectroscopy is a way to probe (or extract) this information.
- Seeing the bulk-edge correspondence.
- Different partitions give access to different types of excitations.
- Entanglement spectroscopy is a concrete tool, requiring only the ground state (example of the fractional Chern insulators).
- What is the meaning of the counting exponential reduction? Efficient description using a matrix product state representation (but this is another story...).
Counting is great!
Conclusion

Counting is great!
(you just have to be careful...)