Spectral properties of reduced fermionic density operators and superselection rules

Sergey N. Filippov

Institute of Physics and Technology, Russian Academy of Sciences and Moscow Institute of Physics and Technology

Abstract

We consider pure fermionic states with varying number of quasiparticles and demonstrate that spectra of partial states (reduced with respect to some bipartition) are not identical as it takes place for bosons and conventional states in spaces with tensor product structure. We fully characterize pure states with equispatial partial states and show that they are related via local unitary operations with states satisfying the parity superselection rule [1]. Thus, valid purifications for fermionic density operators are found. As a byproduct, the positive semidefinite density matrix for fermions is constructed without implying the state gaussianity. The developed formalism clearly indicates that the conventional partial trace methods are not applicable to fermionic systems.

Notation

Consider the algebra of canonical anticommutation relations (CAR algebra) generated by the ladder operators $a_k, a_k^\dagger$ satisfying the relations [2, 3]

$$
a_k a_{k'} + a_{k'} a_k = d_{kk'}, \quad a_k a_{k'} + a_{k'} a_k = 0,
$$

where $k, k' = 1, \ldots, n$. Let $H_n$ be the Hilbert space with the dimension $\dim H_n = 2^n$ and $(\{j_1, \ldots, j_n\})$ is the fixed orthonormal basis, where $j_k = 0, 1$. We shall suppose that $A$ is realized as the algebra of all bounded operators in $H_n$, such that

$$
a_{k_1} \cdots a_{k_n} = \left\{ \begin{array}{ll}
\sum_j \lambda_{k_1 j_1} \cdots \lambda_{k_n j_n} - 1 & \text{if } j_k = 0, j_{k'} = 1 \\
0 & \text{otherwise}
\end{array} \right.
$$

where $\lambda_{k j}$ is a positive real number. In what follows, $1 \leq k, k' \leq n$.

Definition 1 The state $\langle \psi |$ satisfies the parity superselection rule if a unit vector $| \psi \rangle$ in $H$ has the form

$$
\psi = \sum_{j_1, \ldots, j_n \in \{0, 1\}} \lambda_{j_1 \cdots j_n} | j_1, \ldots, j_n \rangle \lambda_{j_1 \cdots j_n}^* \in C \langle \psi |
$$

where the numbers $\sum_j j_k$ are even or odd alternatively for all non-zero $\lambda_{j_1 \cdots j_n}$.

Definition 2 The state $\omega(a_{k_1}, a_{k_2}) = 0$ for any choice of ladder operators $a_{k_1} = a, a_{k_2} = a^\dagger$, $s = 1, \ldots, 2k + 1$.

Proposition 1 The pure state $\omega$ is even if it satisfies the parity superselection rule.

Reduced fermionic density matrices

Let $\rho$ be obtained by joining two algebras of canonical anticommutation relations $A_1$ and $A_2$ with the generators \( a_{1i}, a_{2j}, a_{1i}^\dagger, a_{2j}^\dagger \) and respective partial states of $\rho$ are given by $A = A_1 \otimes A_2$, respectively. Fix a unit vector $| \psi \rangle \in H$.

Corollary 1 The element $\rho$ defines a valid quantum state if it can be represented in the form

$$
\rho = \sum_{k, k' \in \mathcal{J}} \lambda_{kk'} A_k A_{k'},
$$

where $| \lambda_{kk'} \rangle$ is a positive semidefinite matrix with the unit trace $\sum_{k, k' \in \mathcal{J}} | \lambda_{kk'} \rangle = 1$ (fermionic density matrix).

Corollary 2 The $2^n \times 2^n$ matrix with elements $| \lambda_{kk'} \rangle$ is a density matrix of an $n$-mode fermionic state over which the average is taken.

Example. Using Eq. (2) and Corollary 2, the 2-mode fermionic state can be described by the density matrix

$$
\{a_1 a_2 \langle a_1^\dagger a_2^\dagger | \psi \rangle | a_2^\dagger a_1^\dagger \} = \{a_1 a_2 \langle a_1^\dagger a_2^\dagger | \psi \rangle | a_2^\dagger a_1^\dagger \}
$$

Spectra of reduced states

To anticipate general results, let us begin with the simplest case of two fermionic modes that can be occupied by a system with a number of varying fermionic quasiparticles with the state $| \psi \rangle = c_{10}|10 \rangle + c_{11}|11 \rangle$, which is formed from the conventional states

$$
| \psi \rangle |_{\text{pure}} = \frac{1}{\sqrt{2}} (|10 \rangle + |11 \rangle),
$$

The reduced (partial) density matrices of the first and the second mode read

$$
|\langle a_1^\dagger a_1 | \langle a_2^\dagger a_2 | & \langle a_1^\dagger a_2^\dagger | \langle a_2^\dagger a_1^\dagger |
$$

respectively. Note that both matrices (5) cannot be simultaneously obtained from the total density matrix (4) by conventional partial trace methods. This is a peculiar property of fermionic states which differs them from the bosonic ones.

Conclusions

We have fully characterized general pure fermionic states such that their reduced density operators are equispectrally. Equispectrality automatically takes place for bosonic systems and systems of distinguishable fermions. However, it does not necessarily hold true for systems composed of indistinguishable fermionic quasiparticles. On the other hand, equispectrality is a natural quantum information property that reflects the fact that entropies of subsystems must coincide. We have found necessary and sufficient conditions for spectra of partial states to be identical. As a byproduct of our research, we have analyzed the density matrix formalism for general case of fermionic states and provided the explicit construction of density matrices (Corollary 2 and Example).

References


Contacts

E-mail: sergey.filippov@phystech.edu
granos@mi.ras.ru