Exchange statistics on graphs

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1 Quantum statistics

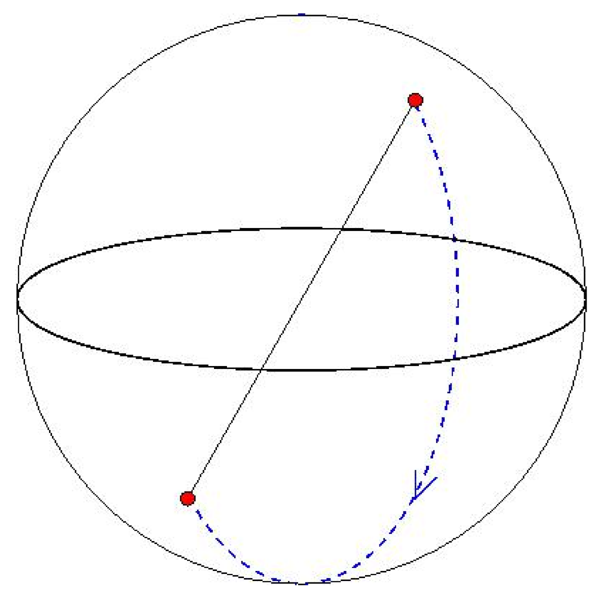
- When two fermions are exchanged the sign of a wave function flips and for bosons it stays the same.
- Leinaas and Myrheim [1]: This postulate can be understood in terms of topological properties of the classical configuration space of indistinguishable particles.
- M – the one-particle classical configuration space. The space of n distinct points in M:

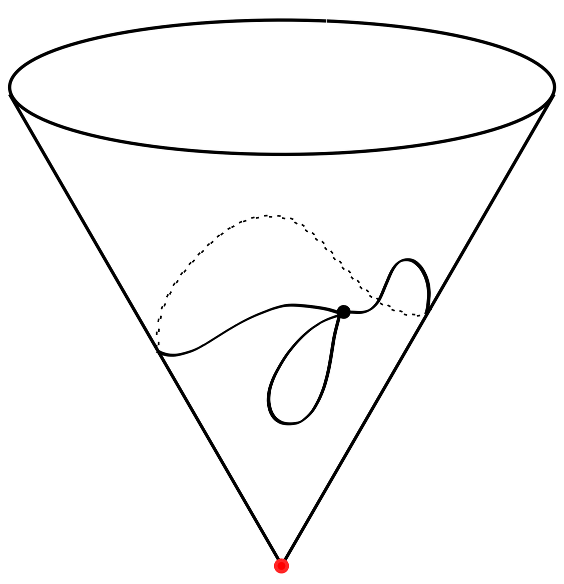
$$F_n(M) = \{(x_1, x_2, \ldots, x_n) : x_i \in X, \, x_i \neq x_j\}.$$

- The configuration space is the quotient space

$$C_n(M) = F_n(M)/S_n.$$

- Any closed loop in $C_n(M)$ represents a process in which particles start at some configuration and end up in the same configuration modulo that they might have been exchanged.
- The space of all loops up to continuous deformations equipped with loop composition is the fundamental group $\pi_1(C_n(M))$.
- The abelianization of the fundamental group is the first homology group $H_1(C_n(M))$, and its structure plays an important role in the characterization of quantum statistics.
- When $M = \mathbb{R}^m$ and $m \geq 3$ we have $H_1(C_n(M)) = \mathbb{Z}_2$ - bosons and fermions.
- When $M = \mathbb{R}^2$ we have $H_1(C_n(M)) = \mathbb{Z}$ - anyon statistics.



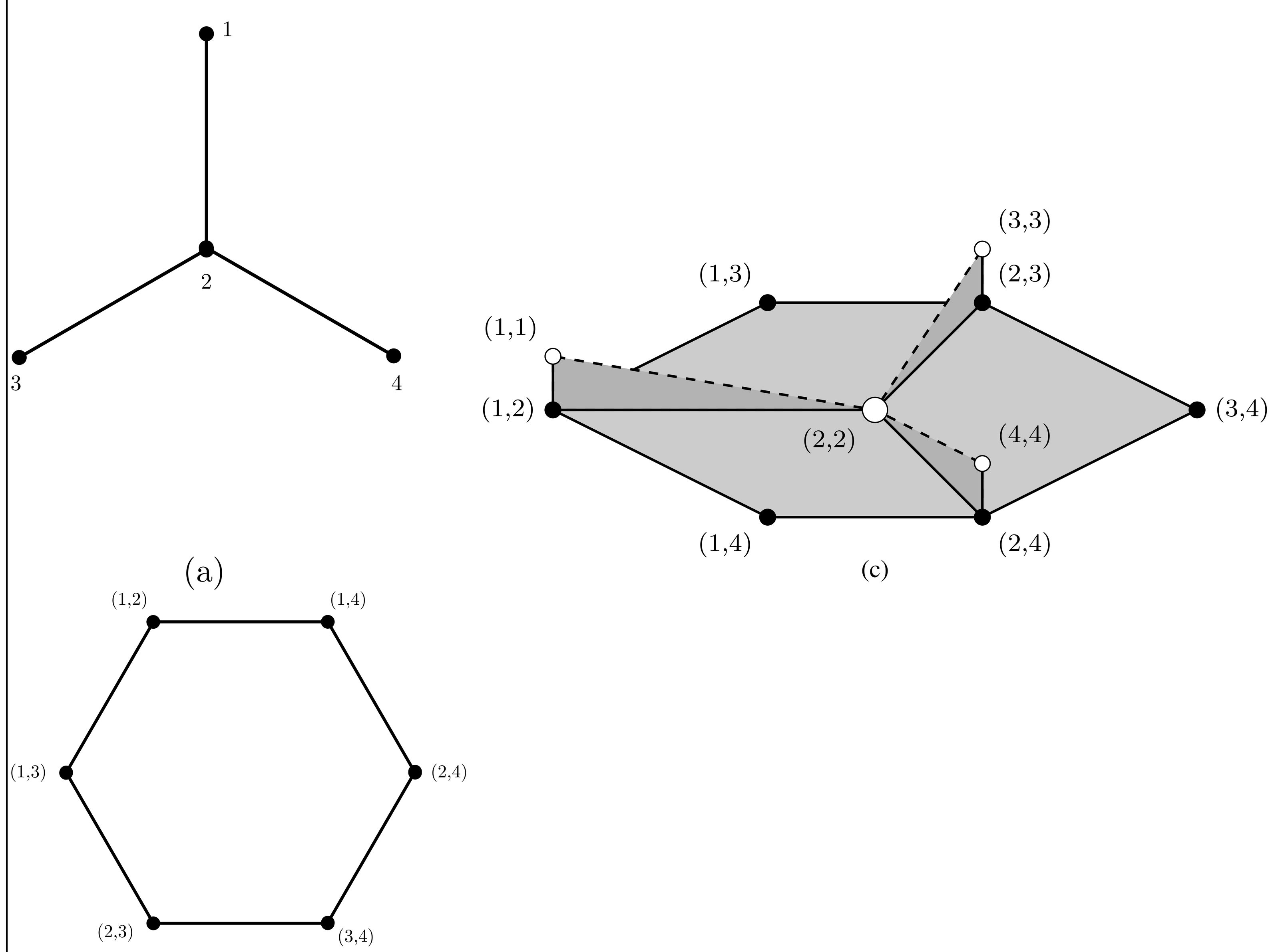


Graph configuration spaces

- We define the n-particle combinatorial configuration space as

$$\mathcal{D}^n(\Gamma) = (\Gamma^{\times n} - \tilde{\Delta})/S_n.$$

- Under certain conditions $\mathcal{D}^n(\Gamma) \hookrightarrow C_n(\Gamma)$ is a homotopy equivalence.
- $\mathcal{D}^n(\Gamma)$ is a cell complex.

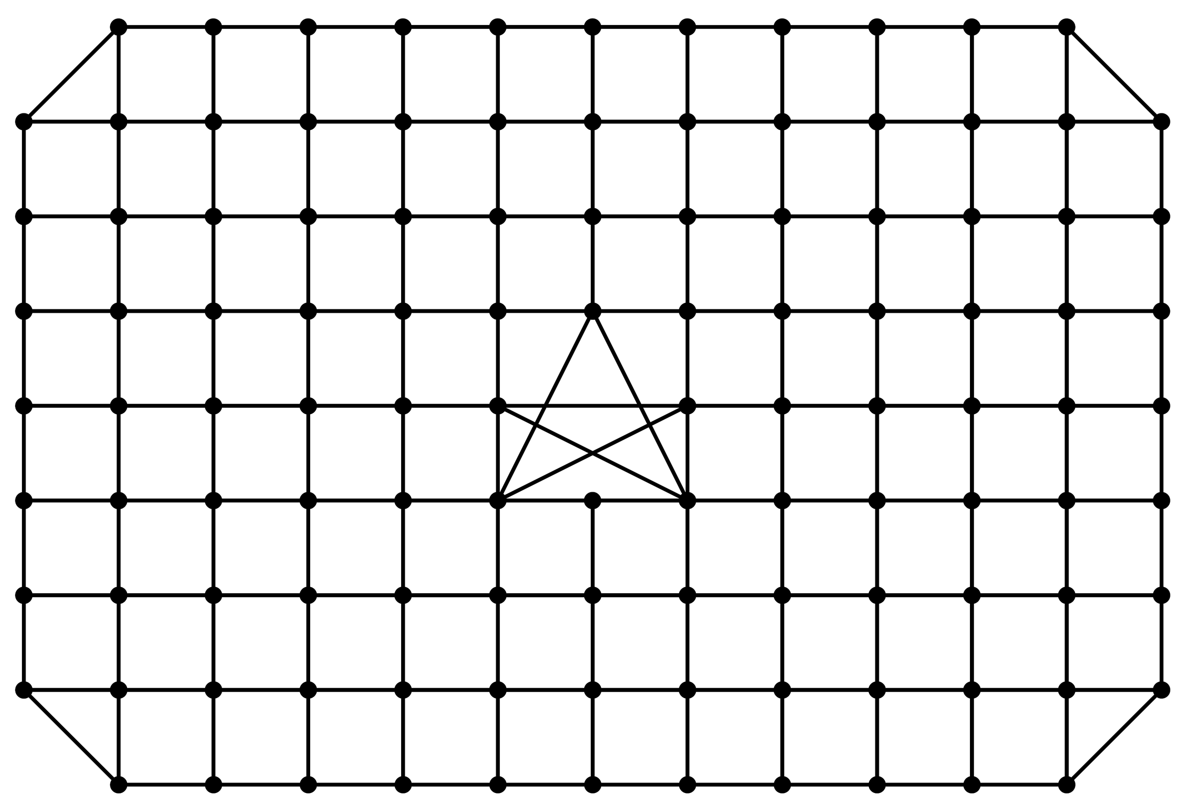


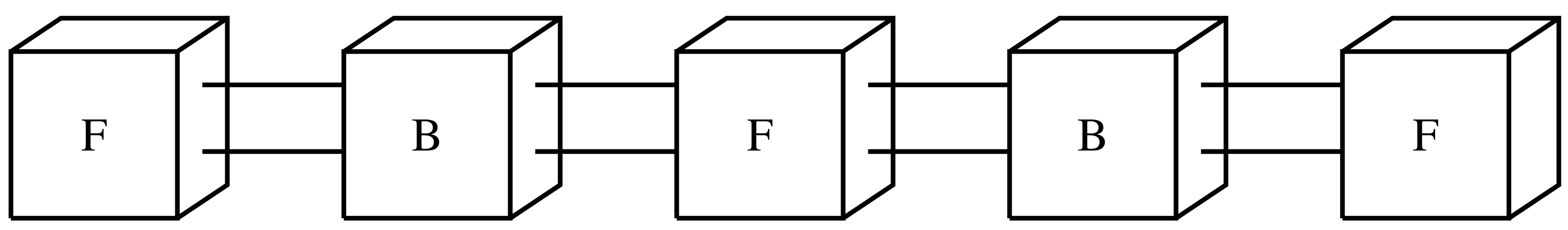
2 Exchange statistics on graphs

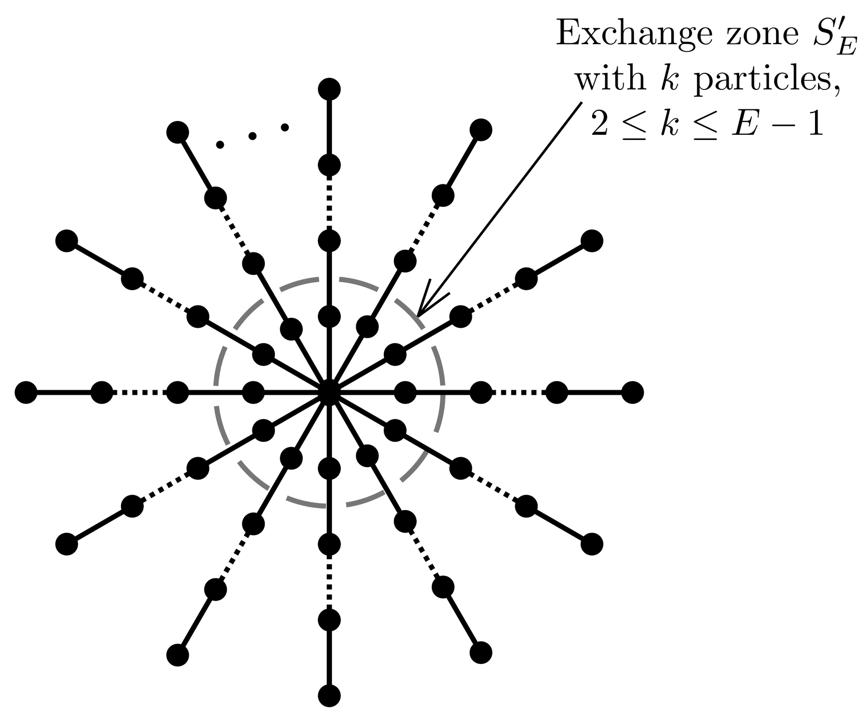
The key topological determinants of the quantum statistics on graphs [2]:

1. The connectivity of a graph.
2. The first homology group $H_1(C_n(\Gamma)) = \mathbb{Z}^{\beta_1} \oplus A$, where β_1 is the number of independent cycles in Γ and A determines quantum statistics.
3. For 1-connected graphs number of anyon phases depends on the number of particles.
4. For 2-connected graphs quantum statistics stabilizes with respect to the number of particles $H_1(C_n(\Gamma)) = H_1(C_2(\Gamma))$.
5. For 3-connected non-planar graphs $A = \mathbb{Z}_2$, i.e. the usual bosonic/fermionic statistics is the only possibility.
6. Planar 3-connected graphs support one anyon phase, $A = \mathbb{Z}$.
7. Fom the quantum statistics perspective, one can say that 3-connected graphs mimic $\mathbb{R}^2$ when they are planar and $\mathbb{R}^3$ when not.

$$H_1(\mathcal{D}^n(\Gamma)) = \mathbb{Z}^{\beta(\Gamma)+N_1+N_2+N_3} \oplus \mathbb{Z}_2^{N_3'}.$$







(a)

$$\binom{n+E-2}{E-1}(E-2) - \binom{n+E-2}{E-2} + 1 \, .$$

[1] Leinaas J M, Myrheim J 1977 On the theory of identical particles. Nuovo Cim.37B, 1–23.
[2] Harrison J M, Keating J P,Robbins J M, Sawicki A 2014 n-particle quantum statistics on graphs, Commun. Math. Phys. 330, 1293-1326