Caroline Terquem

Vacation work: Problem set 0

Revisions

At the start of the second year, you will receive the second part of the Electromagnetism course. This vacation work contains a set of problems that will enable you to revise the material covered in the first year Electromagnetism course.

Some of the problems below are taken from:
Introduction to Electrodynamics, David J. Griffiths, 4th Edition
Electricity and Magnetism, Edward M. Purcell and David J. Morin, 3rd Edition.

Electrostatics

Problem 1: Field and potential from charged ring
A thin ring of radius a carries a charge q uniformly distributed. Consider the ring to lie in the x–y plane with its centre at the origin.

a) Find the electric field $\mathbf{E}$ at a point P on the z–axis.

b) Find the electric potential V at P.

c) A charge $-q$ with mass m is released from rest far away along the axis. Calculate its speed when it passes through the centre of the ring. (Assume that the ring is fixed in place).

Problem 2: Field from charged disc
The ring in the previous problem is replaced by a thin disc of radius a carrying a charge q uniformly distributed. Consider the disc to lie in the x–y plane with its centre at the origin.

a) Find the electric field $\mathbf{E}$ at a point P on the z–axis.

b) Check that the values of $\mathbf{E}$ at $z = 0$ and in the limit $z \gg a$ are consistent with expectations.

Problem 3: Hydrogen atom

According to quantum mechanics, the hydrogen atom in its ground state can be described by a point charge $+q$ (charge of the proton) surrounded by an electron cloud with a charge density $\rho(r) = -C\mathrm{e}^{-2r/a_0}$. Here a_0 is the Bohr radius, 0.53×10^{-10} m, and C is a constant.

a) Given that the total charge of the atom is zero, calculate C.

b) Calculate the electric field at a distance r from the nucleus.

c) Calculate the electric potential, $V(r)$, at a distance r from the nucleus. We give:

$$\int \left(\frac{1}{\alpha r'} + 1\right) \frac{\mathrm{e}^{-\alpha r'}}{r'} dr' = -\frac{\mathrm{e}^{-\alpha r}}{\alpha r}.$$

Problem 4: Energy of a charged sphere

We consider a solid sphere of radius a and charge Q uniformly distributed.

a) Calculate the electric field $E(r)$ and the electric potential $V(r)$ at a distance r from the centre of the sphere.

Find the energy U stored in the sphere three different ways:

b) Use the potential energy of the charge distribution due to the potential $V(r)$:

$$U = \frac{1}{2} \int_{\mathcal{V}} \rho V \, d\tau,$$

where ρ is the charge density and the integral is over the volume $\mathcal{V}$ of the sphere.

c) Use the energy stored in the field produced by the charge distribution:

$$U = \int_{\text{space}} \frac{\epsilon_0 E^2}{2} d\tau,$$

where the integral is over *all space*.

d) Calculate the work necessary to assemble the sphere by bringing successively thin charged layers at the surface.

Problem 5: Conductors

A metal sphere of radius R_1, carrying charge q, is surrounded by a thick concentric metal shell of inner and outer radii R_2 and R_3. The shell carries no net charge.

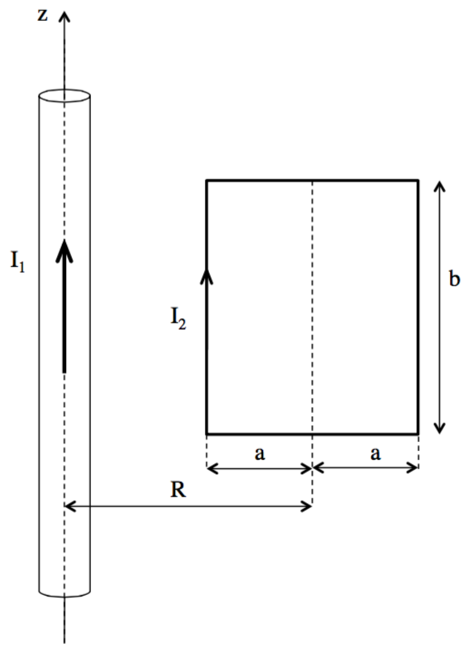
a) Find the surface charge densities at R_1, R_2 and R_3.

b) Find the potential at the centre, choosing $V = 0$ at infinity.

c) Now the outer surface is grounded. Explain how that modifies the charge distribution. How do the answers to questions (a) and (b) change?

Magnetostatics

Problem 6: Force on a loop



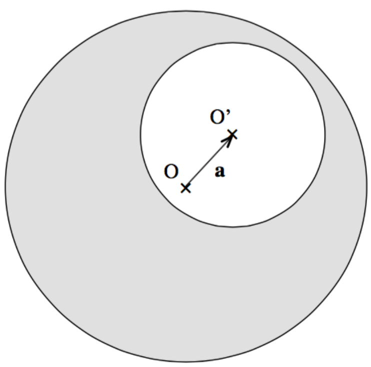
A long thin wire carries a current I_1 in the positive z–direction along the axis of a cylindrical co-ordinate system. A thin, rectangular loop of wire lies in a plane containing the axis, as represented on the figure. The loop carries a current I_2.

a) Find the magnetic field due to the long thin wire as a function of distance r from the axis.

b) Find the vector force on each side of the loop which results from this magnetic field.

c) Find the resultant force on the loop.

Problem 7: Magnetic field in off–centre hole



A cylindrical rod carries a uniform current density J. A cylindrical cavity with an arbitrary radius is hollowed out from the rod at an arbitrary location. The axes of the rod and cavity are parallel. A cross section is shown on the figure. The points O and O' are on the axes of the rod and cavity, respectively, and we note $\mathbf{a} = \mathbf{OO}'$.

a) Show that the field inside a solid cylinder can be written as $\mathbf{B} = (\mu_0 J/2)\hat{\mathbf{z}}\times\mathbf{r}$, where $\hat{\mathbf{z}}$ is the unit vector along the axis and $\mathbf{r}$ is the position vector measured perpendicularly to the axis.

b) Show that the magnetic field inside the cylindrical cavity is uniform (in both magnitude and direction).

Please turn over...

Problem 8: Magnetic field at the centre of a sphere

A spherical shell with radius a and uniform surface charge density σ spins with angular frequency ω around a diameter. Find the magnetic field at the centre.

Problem 9: Motion of a charged particle in a magnetic field

A long thin wire carries a current I in the positive z–direction along the axis of a cylindrical co-ordinate system. A particle of charge q and mass m moves in the magnetic field produced by this wire. We will neglect the gravitational force acting on the particle as it is very small compared to the magnetic force.

a) Is the kinetic energy of the particle a constant of motion?

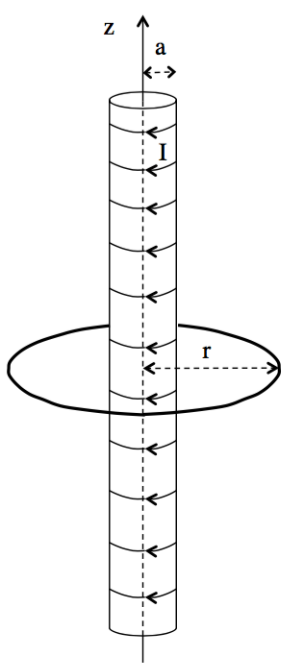
b) Find the force $\mathbf{F}$ on the particle, in cylindrical coordinates.

c) Obtain the equation of motion, $\mathbf{F} = md\mathbf{v}/dt$, in cylindrical coordinates for the particle.

d) Suppose the velocity in the z–direction is constant. Describe the motion.

Electromagnetic induction

Problem 10: Growing current in a solenoid



An infinite solenoid has radius a and n turns per unit length. The current grows linearly with time, according to $I(t) = kt$, $k > 0$. The solenoid is looped by a circular wire of radius r, coaxial with it. We recall that the magnetic field due to the current in the solenoid is $B = \mu_0 nI$ inside the solenoid and zero outside.

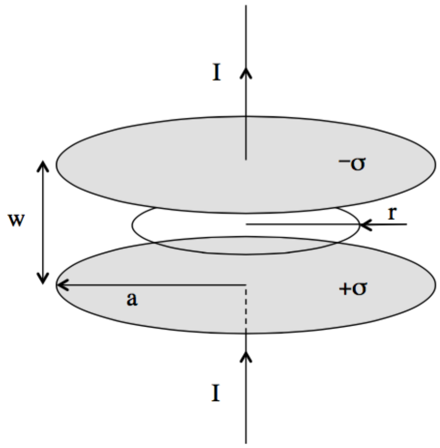
a) Without doing any calculation, explain which way the current induced in the loop flows.

b) Use the integral form of Faraday's law, which is $\oint \mathbf{E} \cdot d\mathbf{l} = -d\Phi/dt$, to find the electric field in the loop for both $r < a$ and $r > a$. Check that the orientation of $\mathbf{E}$ agrees with the answer to question (a).

c) Verify that your result satisfies the local form of the law, $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$.

Maxwell's equations

Problem 11: Energy flow into a capacitor



A capacitor has circular plates with radius a and is being charged by a constant current I. The separation of the plates is $w \ll a$. Assume that the current flows out over the plates through thin wires that connect to the centre of the plates, and in such a way that the surface charge density σ is uniform, at any given time, and is zero at $t = 0$.

a) Find the electric field between the plates as a function of t.

b) Consider the circle of radius $r < a$ shown on the figure (and centered on the axis of the capacitor). Using the integral form of Maxwell's equation $\boldsymbol{\nabla}\times\mathbf{B} = \epsilon_0\mu_0\partial\mathbf{E}/\partial t$ over the surface delimited by the circle, find the magnetic field at a distance r from the axis of the capacitor.

c) Find the energy density u and the Poynting vector $\mathbf{S}$ in the gap. Check that the relation:

$$\frac{\partial u}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{S},$$

is satisfied.

d) Consider a cylinder of radius $b < a$ and length w inside the gap. Determine the total energy in the cylinder, as a function of time. Calculate the total power flowing into the cylinder, by integrating the Poynting vector $\mathbf{S}$ over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the cylinder.

e) When $b = a$, and assuming that we can still neglect edge effects in that case, check that the total power flowing into the capacitor is:

$$\frac{d}{dt}\left(\frac{1}{2}QV\right),$$

where V is the voltage across the capacitor (since $QV/2$ is the energy stored in the electric field in the capacitor).

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Problem set 1

Potentials

Problem 1: Magnetic vector potential

We consider a finite segment of straight wire of length $2L$ carrying a steady current I.

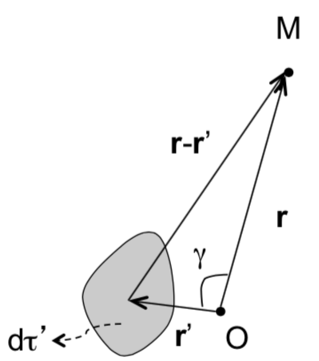
a) Calculate the magnetic vector potential $\mathbf{A}$ at a point P a distance r from the wire along the perpendicular bisector. We give:

$$\int \frac{dz}{\sqrt{z^2+a^2}} = \ln\left(z+\sqrt{z^2+a^2}\right).$$

b) Using $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ and assuming $L \gg r$, calculate the magnetic field at P. Check that your answer is consistent with what is expected from Ampère's law.

Problem 2: Expansion in Legendre polynomials

This problem is useful to understand that a multipole expansion is nothing more than a development in Taylor series of $1/|\mathbf{r}'-\mathbf{r}|$.



The scalar potential V created at a point M by a localised charge distribution is:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\iiint_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}'-\mathbf{r}|}d\tau',$$

where the integration is over the volume $\mathcal{V}$ of the distribution and the position vectors $\mathbf{r}$ and $\mathbf{r}'$ are measured from an origin O chosen arbitrarily.

Assuming $r' \ll r$, expand $1/|\mathbf{r}'-\mathbf{r}|$ in Taylor series up to 3rd order in r'/r. Use the result to give an expression for V in term of Legendre polynomials including the terms up to $l=3$.
We give: $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$, $P_3(x)=\frac{1}{2}(5x^3-3x)$.

Problem 3: Dipole

a) Show that the energy of a physical dipole $\mathbf{p}$ in an electric field $\mathbf{E}$ (not necessarily uniform) is given by $U = -\mathbf{p} \cdot \mathbf{E}$.

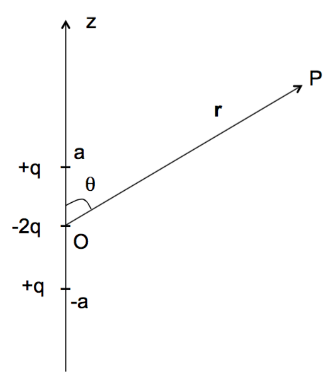
b) We consider two dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ and note $\mathbf{r}$ the position vector of $\mathbf{p}_2$ measured from $\mathbf{p}_1$. Show that the interaction energy of the two dipoles is:

$$U_{\text{int}} = \frac{1}{4\pi\epsilon_0 r^3} \left[\mathbf{p}_1 \cdot \mathbf{p}_2 - 3\left(\mathbf{p}_1 \cdot \hat{\mathbf{r}}\right)\left(\mathbf{p}_2 \cdot \hat{\mathbf{r}}\right)\right],$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$.

c) Draw graphs showing how U_{int} depends upon the relative orientation of the dipoles in the following cases: (i) $\mathbf{p}_1$ is parallel to $\mathbf{r}$, (ii) $\mathbf{p}_1$ is perpendicular to $\mathbf{r}$.

Problem 4: Quadrupole



A system of three charges, aligned along the z–axis, consists of a charge $-2q$ at the origin O and two $+q$ charges at $z = -a$ and $z = a$. We note (r, θ, φ) the spherical coordinates.

a) Find the potential at $r \gg a$ using the multipole expansion with origin at O. Justify why this system of charges is called a quadrupole.
We give: $P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$.

b) Show that there is not translational force or couple on the quadrupole in a uniform electric field.

c) We consider a point charge Q placed at a point P a distance $r \gg a$ away from the quadrupole. Show that the torque about P on the quadrupole due to the charge is $\Gamma = 3Qqa^2 \sin 2\theta/(4\pi\epsilon_0 r^3)$ and find its direction.

d) Find the energy of the quadrupole in the potential of the charge Q. In which direction does the quadrupole rotate under the effect of the torque?

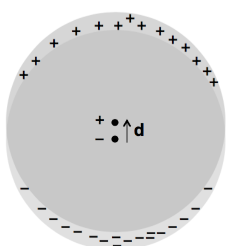
Problem 5: Electric field due to a sphere with surface charge density $\propto \cos\theta$
We consider a sphere of radius R and a spherical coordinate system (r, θ, φ) with origin at the centre of the sphere. The surface of the sphere carries a charge density $\sigma = k\cos\theta$.

a) Calculate the potential V using the fact that it satisfies Laplace's equation both inside and outside the sphere. *Hint*: write the solution of the equation in term of Legendre polynomials and choose $V = 0$ at infinity.

b) Find the electric field $\mathbf{E}$ produced by the sphere.

c) Calculate the electric dipole moment $\mathbf{p}$ of the charge distribution. Check that the electric field outside the sphere calculated in the previous question is that due to the dipole moment $\mathbf{p}$. What can you conclude about the higher multipoles? Express the field inside the sphere as a function of the *polarization* of the sphere, which is the dipole moment per unit volume.

d) A surface charge density $\propto \cos\theta$ can be obtained by superimposing two solid spheres with same radius R, opposite charges Q and $-Q$ uniformly distributed over their volume, and centres slightly apart (separation $d \ll R$), as shown on the figure. From a point outside the spheres, the situation is the same as if the charges were at the centre of the spheres.



Find the relation between Q, $\mathbf{d}$ and $\mathbf{p}$ for the electric field outside the spheres to be the same as in the previous question. Using Gauss's theorem to obtain the contribution from each sphere, calculate the field in the region where the spheres overlap and show that it is the same as in the previous question.

Problem 6: Metal sphere in a uniform electric field (*Pb 5 needs to be done first*)
A metal sphere of radius R with no net charge is placed in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector.

a) Explain qualitatively how the field modifies the charge distribution in the sphere and how this, in turn, affects the electric field.

b) Calculate the electric potential V outside the sphere using the fact that it satisfies Laplace's equation. *Hint*: write the solution of the equation in term of Legendre polynomials and choose $V = 0$ at the surface of the sphere.

c) Using the boundary condition on the electric field, calculate the charge density σ at the surface of the sphere.

d) Using the results from Problem 5, question b, give an expression for the electric field produced by σ. Find the total field inside the sphere. Comment.

Please turn over...

Problem 7: Magnetic field due to a spinning sphere

We consider a sphere of radius R which carries a uniform surface charge density σ and spins with angular velocity ω around a diameter. We use spherical coordinates (r, θ, φ) with origin at the centre of the sphere and the z–axis along the rotation axis.

a) Find the surface current density $\mathbf{K}(\mathbf{r})$ as a function of θ. *Hint:* Consider a ring with a small thickness at the surface of the sphere and perpendicular to the rotation axis.

b) Justify that the magnetic field produced by the surface current can be written under the form $\mathbf{B} = \boldsymbol{\nabla}\Psi$, where Ψ is a scalar function. Show that Ψ satisfies Laplace's equation inside and outside the sphere. Write the boundary conditions on Ψ. **Caution:** Ψ is not continuous at $r = R$ (explain why).

c) Write Ψ in term of Legendre polynomials. Show that the $l = 1$ term in the expansion, with appropriate coefficients, satisfies Laplace's equation inside and outside the sphere with the boundary conditions given above. Justify that this is **the** solution.

d) Find the magnetic field $\mathbf{B}$ produced by the sphere.

e) Consider a ring with a small thickness at the surface of the sphere and perpendicular to the rotation axis. Write the magnetic dipole moment of this ring. Find the total magnetic dipole moment $\mathbf{m}$ of the sphere.

f) Check that the magnetic field outside the sphere calculated in question d is that due to the dipole moment $\mathbf{m}$. What can you conclude about the higher multipoles? Express the field inside the sphere as a function of the *magnetization* of the sphere, which is the dipole moment per unit volume.

Problem 8: Separation of variables in cylindrical coordinates

a) Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no dependence on z.

b) Consider an infinitely long straight wire along the z–axis which carries a uniform line charge λ. Calculate the electric field using Gauss's theorem. Find the potential and check that it is a solution of Laplace's equation found in question a.

c) Consider an infinitely long metal pipe of radius R placed at right angle to a uniform electric field $\mathbf{E}_0$. Using the result from question a and appropriate boundary conditions, find the potential outside the pipe. Calculate the charge density induced on the pipe.

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Problem set 2

Electric and magnetic fields in matter

Some of the problems below are taken from:
Introduction to Electrodynamics, David J. Griffiths, 4th Edition

Electric fields in matter

Problem 1: Capacitance and dielectrics

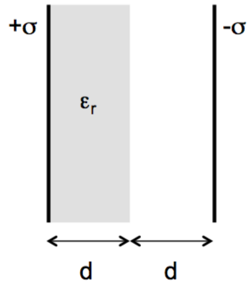
We consider a parallel plate capacitor consisting of two metal surfaces of area A separated by a distance d. We assume $d \ll A^{1/2}$, so that edge effects can be neglected and the electric field can be considered to be uniform between the plates.

a) The capacitor is connected to a battery so that a charge $+Q_0$ is brought from one plate to the other. Using Gauss's law, calculate the electric field E_0 between the plates. Find the potential difference ΔV_0 between the plates and the capacitance C_0. Calculate the potential energy U_0 stored in the capacitor.

b) We now insert a linear dielectric material between the plates while the capacitor remains connected to the battery, which supplies the potential difference ΔV_0. Experimentally, it is found that the charge Q on the plates is increased. Explain why. By representing the system as the superposition of a vacuum capacitor and a polarized dielectric slab, calculate the total electric field E in the capacitor in terms of Q and the polarization P. By substituting $P = \epsilon_0 \chi_e E$, find the relation between the displacement vector $\mathbf{D}$ and Q. Calculate Q in terms of Q_0 and the capacitance C in terms of C_0. Calculate the change in the stored potential energy of the capacitor in terms of U_0.

c) We now come back to question (a) and disconnect the battery before inserting a dielectric material between the capacitor plates. Experimentally, it is found that the potential difference ΔV between the plates decreases. Explain why. Calculate the capacitance C in terms of C_0 and the change in the stored potential energy of the capacitor in terms of U_0.

Problem 2: Capacitor half–filled with a dielectric



We insert a slab of linear dielectric material of dielectric constant ϵ_r and thickness d next to the positive plate in a parallel plate capacitor. The distance between the plates is $2d$, their area is A and they carry a (free) charge density $\pm\sigma$.

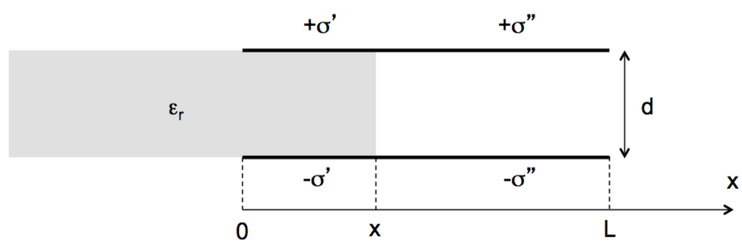
a) Find the electric displacement vector **D**, the electric field **E** and the polarization **P** in each region.

b) Find the potential difference between the plates and the capacitance C. Show that the system can be regarded as two capacitors connected in series.

c) Find the location and amount of all bound charge. Using the distribution of free and bound charges, recalculate the electric field **E** in each region.

Problem 3: Force on a dielectric

We insert a portion of a slab of linear dielectric material of dielectric constant ϵ_r and thickness d on the left hand side of a parallel plate capacitor consisting of two conducting plates of length L, width w and thickness d.



a) We connect the capacitor to a battery to charge the plates, and then disconnect the battery. The total charge on each plate then remains constant equal to $\pm Q$, corresponding to surface charge densities $\pm\sigma'$ and $\pm\sigma''$ on the left and right hand sides, respectively. Using Gauss's law, write the electric fields E' and E'' on the left and right hand sides in terms of σ' and σ''. Find the relation between σ' and σ''. Using the fact that Q is constant, write σ'' in terms of x and the different constants. Find the potential difference between the plates, and then the stored potential energy U of the system, in terms of x. Write the relation between the change in energy, dU, when the dielectric is pulled out a distance dx, and the electric force F exerted by the plates on the dielectric. Calculate F in terms of x. Does this force pull the dielectric into the capacitor or push it out?

b) We now repeat the experiment but leave the capacitor connected to the battery, which supplies the potential difference ΔV. Calculate U in terms of x and the different constants. Write the relation between dU and F when the dielectric is pulled out a distance dx. Make sure to include *all* contributions to the work done in the system. Calculate F in terms of x and show that it is the same as in question (a).

c) The result obtained above can be verified experimentally. However, we have assumed in our calculation that the electric field was uniform between the plates and zero outside. If that were the case in reality, would there be a force on the dielectric? Where does this force come from, and how is it possible that we obtain the right answer given the simplified model we have used?

Problem 4: Sphere with a frozen–in polarization

A sphere of radius R made of a dielectric material carries a frozen–in polarization $\mathbf{P}(\mathbf{r}) = k\mathbf{r}$, where k is a constant and $\mathbf{r}$ is the position vector measured from the centre of the sphere. There are no free charges anywhere.

a) Calculate the bound charges.

b) Using Gauss's law, find the electric field $\mathbf{E}$ inside and outside the sphere.

c) Using the expression of $\mathbf{E}$ and $\mathbf{P}$, find the electric displacement vector $\mathbf{D}$ inside and outside the sphere. Check that it satisfies Gauss's law. Is the dielectric linear?

Problem 5: Electric field within a cavity inside a dielectric

The electric field inside a large piece of dielectric is $\mathbf{E}_0$ and the polarization is $\mathbf{P}$, so that the displacement vector is $\mathbf{D} = \epsilon_0\mathbf{E}_0 + \mathbf{P}$. A cavity is hollowed out of the material. It is small enough that the field and the polarization can be taken as uniform within it. We also assume that the polarization in the dielectric is frozen–in, so that it does not change when the cavity is hollowed out. Calculate, in the cavity, the field in terms of $\mathbf{E}_0$ and $\mathbf{P}$ and the displacement in terms of $\mathbf{D}_0$ and $\mathbf{P}$ in the following cases:

a) The cavity is a small sphere [Hint: Use the superposition principle and, to calculate the electric field inside a uniformly polarized sphere, use the results of Problem 5 in Problem set 1],

b) The cavity is shaped like a long thin needle parallel to $\mathbf{P}$,

c) The cavity is a thin, circular wafer perpendicular to the polarization.

Magnetic fields in matter

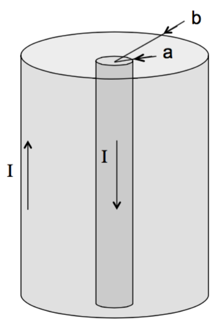
Problem 6: Cylinder with a frozen–in magnetization

An infinitely long cylinder of radius R made of a magnetic material carries a frozen–in magnetization $\mathbf{M} = kr\hat{\mathbf{z}}$, where r is the distance from the axis of the cylinder, $\hat{\mathbf{z}}$ is the unit vector along the axis and k is a constant. There is no free current anywhere. Find the magnetic field $\mathbf{B}$ inside and outside the cylinder by two different methods:

a) Locate all the bound currents and calculate the field they produce,

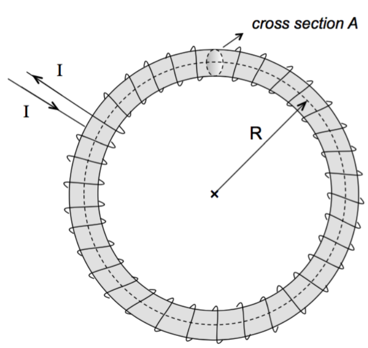
b) Use Ampère's law for $\mathbf{H}$ and the relationship between $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{M}$ to derive $\mathbf{B}$.

Problem 7: Magnetic field in a coaxial cable



A long coaxial cable of inner radius a and outer radius b is filled with an insulating material of magnetic susceptibility χ_m. A current I flows down the inner conductor and returns along the outer one, uniformly distributed over the surfaces. Find the magnetic field $\mathbf{B}$ in the magnetic material between the conductors. As a check, calculate the magnetization $\mathbf{M}$ and the bound currents, and confirm that, together with the free currents, they generate the correct field.

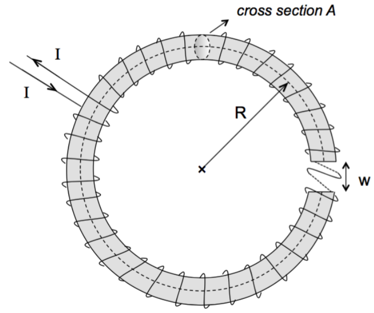
Problem 8: Air gap in an inductor



We consider a toroidal core made of an iron alloy with cross sectional area A and radius $R \gg A^{1/2}$. We make an inductor from the core by wrapping around it N turns of a wire carrying a current I, to use it as an energy storage device.

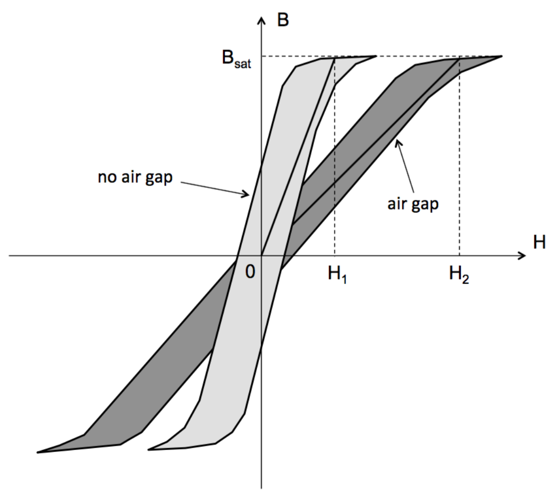
We increase the current from 0 until the material saturates, which happens when the magnetic field B inside it reaches the value $B_{\rm sat}$. Before saturation is reached, the material can be regarded as being linear with relative permeability μ_r.

a) Calculate the magnetic field B in the core and the inductance L. Find the maximum current I_1 that the inductor can carry before the core saturates. Calculate the magnetic energy W_1 that is stored in the inductor when the current reaches the value I_1. If I is increased beyond I_1, how does the inductance vary?



b) To increase the energy that can be stored in the inductor, we have to find a way of increasing the saturation current while keeping L constant. This can be done by introducing an air gap of width w into the core. We take $w \ll R$, so that we can neglect magnetic fringing, that is to say the bulging out of the magnetic field lines as they enter air from the magnetic material.

Calculate the magnetic field B in the core and the number of turns of wire we now have to wrap around the core to keep L constant. Find the maximum current I_2 this second inductor can carry before the core saturates and compare it with I_1. Calculate the energy W_2 stored in the magnetic field at the point of saturation and compare it with W_1. Compare the energy stored in the magnetic material with that stored in the air gap. For numerical applications, use $R = 10$ cm, $w = 3$ mm and $\mu_r = 1500$.



c) Some of the energy stored in the inductor can be recovered by decreasing the current back to 0. The figure shows the $B-H$ curves (hysteresis loops) measured on a toroidal core without and with gap (the straight lines inside the loops are the magnetization curves).

Show on these curves how much energy is released per unit volume by the magnetic material in the core after the current is returned to 0. Comment.

d) Explain what would be the advantages of using iron, and especially *soft* iron, to make magnetic cores. Iron, however, has a major disadvantage when the inductor is used with alternating currents. Can you think of what it is?