Part A: Mathematics for Physics [50 Marks]

1. If $p = \sqrt{3}$ and $q = \sqrt{2}$ evaluate

$$\sqrt{(5p-4q)^2-(4p-5q)^2}\,.$$

[3]

$$= \left(25p^2 - 40pq + 16q^2 - (16p^2 - 40pq + 25q^2)\right)^{1/2}$$

$$= \left(9p^2 - 9q^2\right)^{1/2}$$

$$= 3\left(p^2-q^2\right)^{1/2}$$

$$= 3 \quad \text{if} \quad p^2 = 3,\ q^2 = 2.$$

2. Find the set of real numbers λ for which the quadratic equation

$$x^2 + (\lambda - 3)x + \lambda = 0$$

has distinct, real roots for x. [4]

roots are $x = \dfrac{3-\lambda \pm \sqrt{(\lambda-3)^2 - 4\lambda}}{2}$ (i.e $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\therefore$ distinct, real roots if $(\lambda-3)^2 > 4\lambda$.

solve for $(\lambda-3)^2 = 4\lambda$

$$\lambda^2 - 6\lambda + 9 = 4\lambda$$

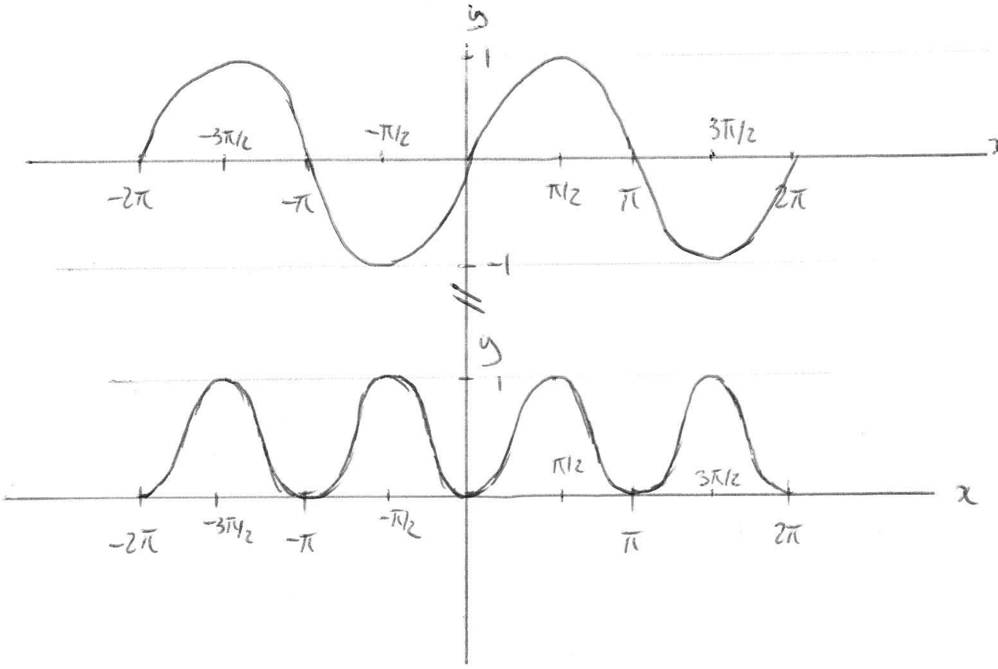
$$\lambda^2 - 10\lambda + 9 = 0$$

$$(\lambda-1)(\lambda-9) = 0 \quad \text{so} \quad \lambda = 1, 9.$$

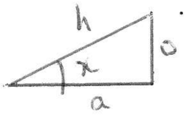
require $(\lambda-3)^2 > 4\lambda$

$\therefore$ $\lambda < 1$ or $\lambda > 9$.

3. (i) Draw sketches of the functions $\sin x$ and $\sin^2 x$ over the range $-2\pi < x < 2\pi$. (Label the axes). [2]



(ii) Explain why, for the range $0 < x < \pi/2$, $\sin x$ is smaller than $\tan x$. [2]



$\sin x = \frac{o}{h}$ $\quad \tan x = \frac{o}{a}$

By Pythagoras' Theorem $h^2 = o^2 + a^2$ $\quad \therefore h > a$

$\therefore \sin x < \tan x$

(iii) Using the equality $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ or otherwise, express $\cos^4\theta$ in terms of $\cos 2\theta$ and $\cos 4\theta$. [3]

$$\cos^4\theta = (\cos^2\theta)^2 = \frac{1}{4}(1 + \cos 2\theta)^2$$

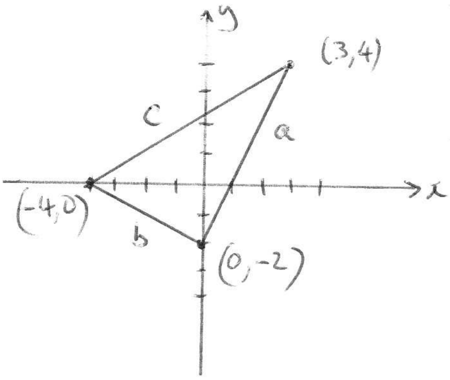
$$= \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta)$$

$$= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)$$

$$= \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$$

4. Show that $(3, 4)$, $(-4, 0)$ and $(0, -2)$ are the vertices of a right-angled triangle, and find its area. [5]



Right angled triangle if $c^2 = a^2 + b^2$

$a^2 = 3^2 + (4+2)^2 = 9 + 36 = 45$

$b^2 = 4^2 + 2^2 = 16 + 4 = 20$

$c^2 = (4+3)^2 + 4^2 = 7^2 + 4^2 = 49 + 16 = 65$

$\therefore$ $a^2 + b^2$ does equal c^2 and so this is a right angled triangle.

Area $= \frac{1}{2} ab = \frac{1}{2} \sqrt{45} \cdot \sqrt{20}$

$= \frac{1}{2} \sqrt{900} = \frac{30}{2} = \underline{\underline{15}}$

5. Find the value of x for which

(i) $\log_2 x = 2$, [1]

$x = 2^2 = 4$

(ii) $\log_x 2 = 2$, [1]

$x^2 = 2 \quad \therefore \quad x = \sqrt{2}$

(iii) $\log_2 2 = x$. [1]

$\log_2 2 = 1 \quad \therefore \quad x = 1$

6. Evaluate $(2.002)^6$ to 4 decimal places. [4]

$$(2.002)^6 = 2^6(1.001)^6 \qquad (1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\dots$$

$$= 64\left(1+0.006+15\times10^{-6}+\dots\right)$$

$$\begin{array}{r} {}^{2}64 \\ 6 \\ \hline 384 \end{array}$$

$$= 64 + 0.384 + 0.00096$$

$$= 64.38496\dots$$

$$\begin{array}{r} {}^{2}64 \\ 15 \\ \hline 640 \\ 320 \\ \hline 960 \end{array}$$

$$= 64.3850 \text{ to 4 decimal places.}$$

7. A ball is dropped vertically from a height h onto a flat surface. After the n^{th} bounce it returns to a height $h/(3^n)$. Find the total distance travelled by the ball. [4]

Distance travelled $d = h + 2\sum_{n=1}^{\infty}\frac{h}{3^n}$

$$= h\left(1+\underbrace{2\sum_{n=1}^{\infty}\left(\tfrac{1}{3}\right)^n}_{\text{geometric series}}\right)$$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$= h\left(1+\frac{2}{3}\sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)^n\right)$$

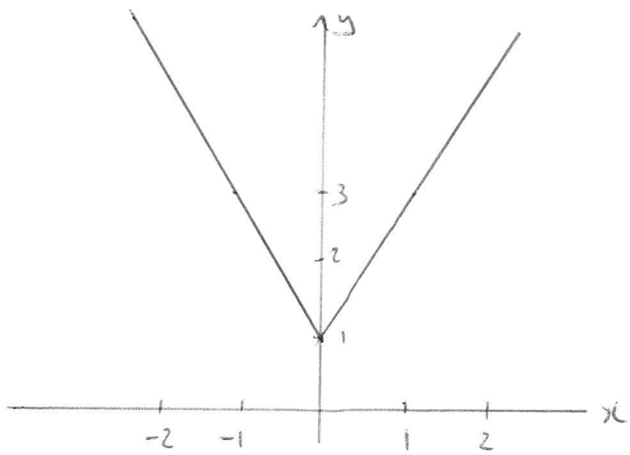
$$= h\left(1+\frac{\frac{2}{3}}{1-\frac{1}{3}}\right)$$

$$= h(1+1)$$

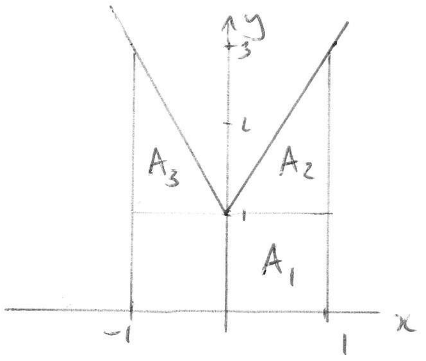
$$= \underline{\underline{2h}}$$

8. (i) Sketch the curve $y = 2\,|x|\,+1$ for $-1 \leq x \leq 1$. [2]

x	y
-1	3
0	1
1	3



(ii) Find the area between the curve $y = 2\,|x|\,+1$, the x-axis, and the lines $x = 1$ and $x = -1$. [2]



$A_1 = 2$

$A_2 = \frac{1}{2}$ ~~base x~~ base x height $= 1$

$A_3 = A_2$

$A = A_1 + A_2 + A_3$

Area $= 2 + 2 = \underline{\underline{4}}$

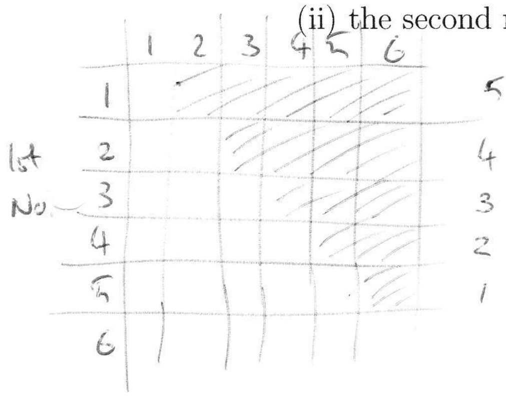
9. Two identical dice are thrown, one after the other. What are the probabilities that
(i) the total of the numbers shown is 6, [2]

Combinations that sum to 6 are: (1,5), (5,1), (2,4), (4,2), (3,3), (3,3)
i.e. 5 combinations out of a total of $6 \times 6 = 36$

$\therefore P = \cancel{6}5/36.$

$= \underline{\underline{1/6}}$

(ii) the second number is greater than the first? [4]



combinations for which 2nd number > 1st

$= 5+4+3+2+1 = 15$

total number of combinations = 36

$\therefore P = \frac{15}{36} = \underline{\underline{\frac{5}{12}}}$

10. A geometric progression and an arithmetic progression have the same first term. The second and third terms of the geometric progression (which are distinct) are equal to the third and fourth terms of the arithmetic progression respectively.
(i) Find the common ratio of the geometric progression. [2]

n	GP	AP
1	a	a
2	ar	$a+d$
3	ar^2	$a+2d$
4	ar^3	$a+3d$
5	ar^4	$a+4d$

$ar = a+2d$

$ar^2 = a+3d$

$\therefore d = \frac{ar-a}{2} = \frac{ar^2-a}{3}$

$\therefore 3ar - 3a = 2ar^2 - 2a \Rightarrow 2r^2 - 3r + 1 = 0$

$r = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} = \frac{1}{2}, 1$

if $r = 1/2$, $d = -a/4$

if $r = 1$, $d = 0$ $\therefore \underline{\underline{r = 1/2}}$

(ii) Show that the fifth term of the arithmetic progression is zero. [3]

if $r = 1/2$, then $d = -a/4$

$\therefore$ 5th term of AP is $a + 4d$

$= a + 4 \cdot \frac{-a}{4}$

$\underline{\underline{= 0}}$

11. What are the largest and smallest values of $y = x^3 - 12x + 1$ for values of x in the range -3 to $+5$? [5]

see if there are max / min in range

$$\frac{dy}{dx} = 3x^2 - 12 = 0$$

$$x^2 = 4 \qquad x = \pm 2$$

$$\frac{d^2y}{dx^2} = 6x$$

at $x = 2$, $\frac{d^2y}{dx^2} = 6 > 0$ $\therefore$ minimum

$$y = 2^3 - 24 + 1 = 8 - 24 + 1$$

$$= \underline{\underline{-15}}$$

at $x = -2$ $\frac{d^2y}{dx^2} = -6 < 0$ $\therefore$ maximum.

$$y = -2^3 + 24 + 1 = -8 + 24 + 1$$

$$= \underline{\underline{17}}$$

check end points of range

$$y(-3) = -27 + 36 + 1 = 10$$

$$y(5) = 125 - 60 + 1 = 66.$$

$\therefore$ $y_{max} = 66$

$\underline{\underline{y_{min} = -15}}$

Part B: Physics [50 Marks]

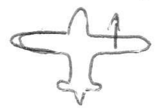
Multiple choice (10 marks).

Please circle **one** answer to each question only.

12. A radioactive source is placed 6 cm from a radiation detector sensitive to all forms of ionizing radiation, and records 74 counts/minute. When a 1 cm thick aluminium plate is placed in the gap then the count rate falls to 45 counts/minute. If the source is removed entirely then the count rate remains the same. If the aluminium plate is removed and the source is placed 2 cm from the detector, the count rate rises to 5000 counts/minute. What forms of radiation does the source emit?

   A alpha only **B** beta only
   C alpha and beta **D** alpha and gamma [1]

13. An aeroplane has two engines, one under each wing. Suppose the left engine stops working. If the aeroplane's controls are not changed then the aeroplane will



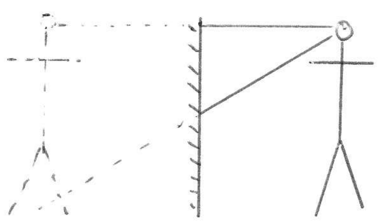
   A slow down and fall but not turn
   B turn right and fall but not slow down
   C turn left, fall and slow down
   D turn right, fall and slow down [1]

14. A sky-diver jumps out of an aeroplane. Which of the following statements is true *after* she reaches terminal velocity?

   A The force of air resistance is the same size as her weight.
   B The force of air resistance is larger than her weight.
   C The force of air resistance is smaller than her weight.
   D She begins to slow down [1]

15. What is the minimum length of a plane mirror in order for you to see a full view of yourself?

   A 1/2 your height **B** 1/4 your height
   C 3/4 your height **D** your full height [1]



16. A metal disc with a hole in it is heated and expands by 1%. The size of the hole has

A increased
B decreased
C stayed the same
D it depends on where the hole is [1]

17. A baby's bath should be at a temperature of 37°C. The bath already contains 10 kg of water at 15°C. Approximately how much hot water, with a temperature of 50°C, should be added to achieve the desired temperature?

A 10 kg B 15 kg
C 17 kg D 20 kg [1]

18. A stone is thrown straight upwards. At the top of its path its velocity is momentarily zero. What is the magnitude of its acceleration at this point?

A $0\,\mathrm{ms^{-2}}$ B $5\,\mathrm{ms^{-2}}$
C $10\,\mathrm{ms^{-2}}$ D $20\,\mathrm{ms^{-2}}$ [1]

19. A battery is connected first across one bulb and then across two bulbs in series. If all the bulbs are identical then the battery delivers

A less current to the series combination
B more current to the series combination
C a lower potential difference across the series combination
D the same current in the two situations [1]

20. An astronaut on the Moon observes a solar eclipse; at the same time the earth experiences a

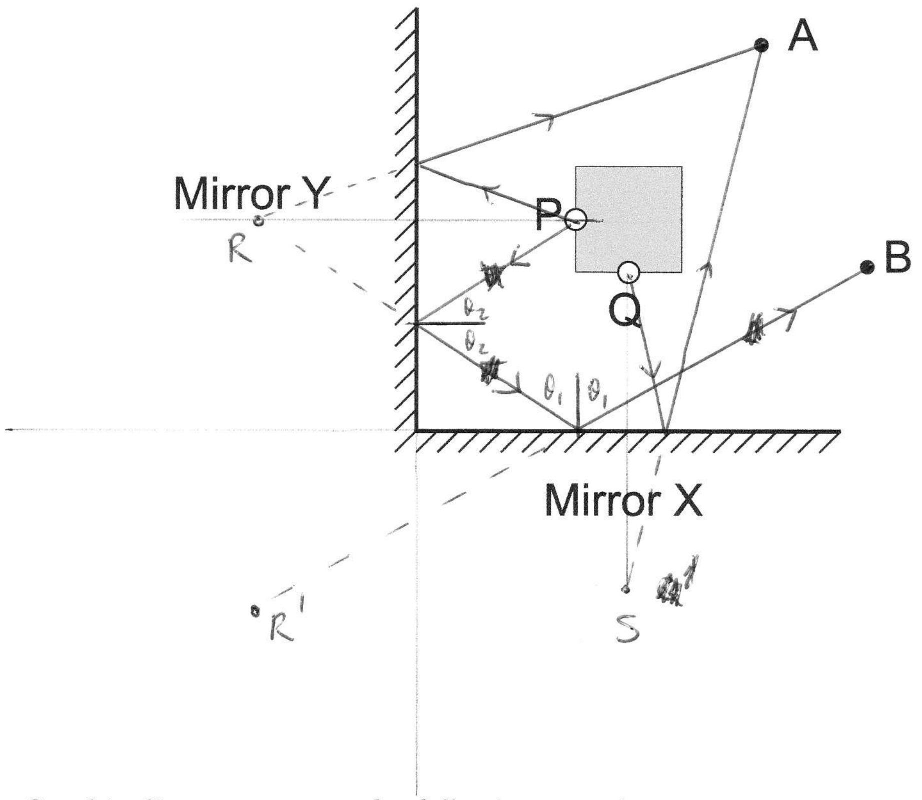
A solar eclipse B lunar eclipse
C both D neither [1]

21. If you run at $5\,\mathrm{ms^{-1}}$ towards a plane mirror, at what speed does your image approach you?

A $5\,\mathrm{ms^{-1}}$ B $2.5\,\mathrm{ms^{-1}}$
C $10\,\mathrm{ms^{-1}}$ D $15\,\mathrm{ms^{-1}}$ [1]

Written answers (20 marks)

22. The diagram below shows two mirrors X and Y, and a solid object with white spots at P and Q.



On this diagram answer the following questions.

(a) An observer at A sees an image of P reflected in mirror Y. Mark R, the position of this image, and draw a ray from P to the observer at A. [2]

(b) In which mirror would an observer at A see an image of spot Q? Mark S, the position of this image. [2]

Mirror X

(c) An observer at B can see an image of P resulting from reflections at *both* mirrors. Draw a ray of light from P to B which enables this image to be seen. [2]

23. Suppose that three new particles are discovered, called the slepton, the hozon and the elephoton. These cannot be investigated on their own, but the properties of certain combinations have been measured. Each particle has its own antiparticle, with the same mass but opposite charge. Given that the following observations are made

- The combination of two sleptons and a hozon has no overall charge
- The combination of three sleptons, a hozon and an elephoton has a charge of +1 q
- The total mass of the three particles is the same as that of six elephotons
- The combination of an anti-slepton and an elephoton has the same mass as three elephotons and a total charge of −2 q

work out the charge (in units of q) and the mass (in units of the elephoton mass) of each of the three particles. [5]

Charge

1 $2s + h = 0 \Rightarrow h = -2s$

2 $3s + h + e = q$

3 $-s + e = -2q$

subst for h in 2

4 $s + e = q$

3−4 $-2s = -3q$

$s = \frac{3q}{2}$

$\therefore h = -3q$

$e = -2q + s$

$= -\frac{q}{2}$

Mass

$M_s + M_h + M_e = 6M_e$

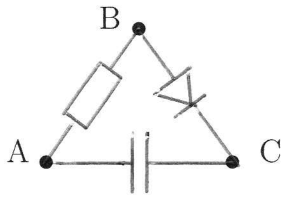
$M_s + M_e = 3M_e$

$\therefore M_h = 3M_e$

$M_s = 2M_e$

	mass/m_e	Charge/q
slepton	2	3/2
hozon	3	−3
elephoton	1	−1/2

24. A black box has three electrical connections labelled A, B, and C, arranged in a triangle as shown below.



The box contains three components: a resistor, a small capacitor and a diode. You know that one component is connected between each pair of terminals, but you cannot see exactly how they are arranged. You make the following observations with a 9 V battery connected in series with an ammeter

- When the battery is connected with + to A and – to B a current of 3 mA flows resistor
- When the battery is connected with + to B and – to C a very large current flows
- When the battery is connected with + to C and – to A no current is measured
- When the battery is connected with – to B and + to C no current is measured

(a) On the diagram sketch the arrangement of the three components in the box indicating the terminals clearly. [4]

(b) Calculate the resistance of the resistor. [2]

$$R = \frac{V}{I} = \frac{9}{3\times10^{-3}} = 3\times10^{3} = 3\,k\Omega.$$

(c) What would happen if the battery were connected with – to A and + to B? [1]

Would get a current of 3mA again

(d) What would happen if the battery were connected with – to C and + to A? [2]

Current would flow through resistor and then through diode. current will again be ~ 3mA, but slightly less due to small voltage drop across the diode

Long question (20 marks)

25. Consider an elevator car (lift) with a mass of 700 kg which can carry up to 600 kg of passengers. Use $g = 10\,\mathrm{m\,s^{-2}}$.

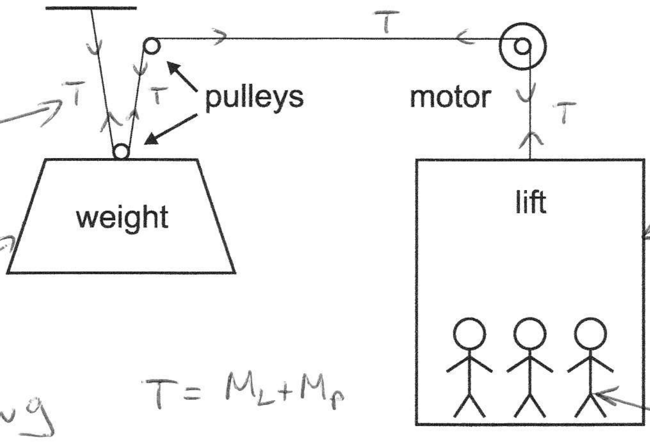
    (a) An electric motor is used to raise the elevator with a full load from the ground floor to the third floor, 9 m higher, in 30 s. Calculate the total energy needed and the power of motor required. [4]

$$E = mgh = (700 + 600) \times 10 \times 9 = 1300 \times 90 = \underline{\underline{117000\ \mathrm{J}}}$$

$$\text{Power} = \frac{E}{t} = \frac{117000}{30} = 1300 \times 3 = \underline{\underline{3900\ \mathrm{W}}}$$

    (b) The elevator system can be made more efficient by incorporating a counterweight, with a mass of 2000 kg as shown in the diagram below. What total mass of people is needed in the car to balance the system? [2]

Pulley system gives mechanical advantage



$M_L = 700$ Kg

Mass M_W

M_P.

$$2T = M_W g \qquad T = M_L + M_P$$

$$\therefore \frac{M_W}{2} = 700 + M_P$$

$$M_W = 2000\ \mathrm{Kg}$$

$$\therefore 1000 = 700 + M_P$$

$$\underline{\underline{M_P = 300\ \mathrm{Kg}}}$$

(c) When the system is balanced the elevator takes 30 s to make the upwards journey to the third floor. Calculate the average speed and the kinetic energy of the system at this speed. [4]

Average speed of elevator $= \frac{9}{30} = \frac{3}{10}\ ms^{-1}$ $\left(\frac{9\,m \text{ in}}{30s}\right)$

Counterweight moves 1/2 as far and so has average speed $= \frac{3}{20}\ ms^{-1}$

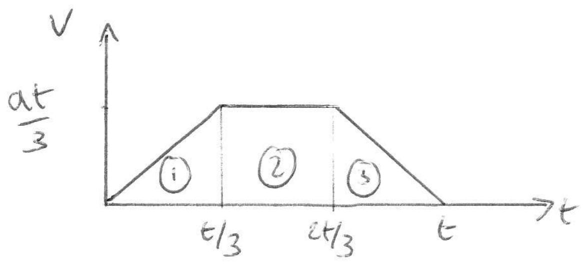
$K.E = \frac{1}{2}mv^2$

$KE_{ELEVATOR} = \frac{1}{2} \times 1000 \times \left(\frac{3}{10}\right)^2 = 45\ J$

$KE_{COUNTERWEIGHT} = \frac{1}{2} \times 2000 \times \left(\frac{3}{20}\right)^2 = \frac{45}{2}\ J$

$\therefore$ Total K.E. $= \underline{\underline{67.5\ J}}$

(d) The elevator takes 10 s to accelerate and 10 s to decelerate using the same magnitude of constant force in each case. Calculate the greatest speed which the lift attains. [4]



$V_{max} = \frac{at}{3}$ where $t = 30s$.

~~S~~ $s = ut + \frac{1}{2}at^2$

$S_1 = \frac{1}{2}a\left(\frac{t}{3}\right)^2$

$S_2 = \frac{at}{3} \cdot \frac{t}{3}$

$S_3 = \frac{at}{3} \cdot \frac{t}{3} - \frac{1}{2}a\left(\frac{t}{3}\right)^2$

$\therefore S_{TOT} = S_1 + S_2 + S_3 = \frac{2at^2}{9}$

$\therefore V_{max} = \frac{at}{3} = \frac{3S}{2t} = \frac{3 \times 9}{2 \times 30} = \frac{9}{2 \times 10} = \frac{9}{20} = \underline{\underline{0.45\ ms^{-1}}}$

$a = \frac{3V_{max}}{t} = \frac{3 \times 0.45}{30} = \underline{\underline{0.045\ ms^{-2}}}$

(e) A man has a mass of 68 kg. While standing in the balanced lift, he measures his weight (in N) using spring scales as the lift moves up as described in part (d). Calculate (i) the reading on the scales when the lift is stationary, (ii) the change in this reading when the lift is accelerating upwards, and (iii) the reading on the scales when the lift is moving upwards at the steady maximum speed calculated above. [6]

i) $W = mg \qquad g = 10\,ms^{-2},\ m = 68\,kg$

$\therefore W = 680\,N.$

ii) Now feel extra acceleration of lift

$\therefore W = m(g + a)$

$= 680 + 68 \times 0.045$ (acceleration calculated in previous section.)

$$\begin{array}{r} 68 \\ 45 \\ \hline 2720 \\ 340 \\ \hline 3060 \end{array}$$

$45 \times 68 = 3060$

$68 \times 0.045 = 3.06$

$\therefore W = 683.06\,N.$

extra weight $\simeq \underline{\underline{3N}}$

iii) at steady maximum speed, no extra acceleration.

$\therefore W = 680\,N$ again

i.e. same as when stationary.